

ABOUT THE ALGEBRAIC CLOSURE OF THE FIELD OF POWER SERIES IN SEVERAL VARIABLES IN CHARACTERISTIC ZERO

GUILLAUME ROND

ABSTRACT. We construct algebraically closed fields containing an algebraic closure of the field of power series in several variables over a characteristic zero field. Each of these fields depends on the choice of an Abhyankar valuation and are constructed via the Newton-Puiseux method. Then we study more carefully the case of monomial valuations and we give a result generalizing the Abhyankar-Jung Theorem for monic polynomials whose discriminant is weighted homogeneous. Essentially this result asserts that the Galois group of such a polynomial is isomorphic to the Galois group of one weighted homogeneous polynomial.

CONTENTS

1. Introduction	1
2. Notations	5
3. Homogeneous elements with respect to an Abhyankar valuation	7
4. Newton method and algebraic closure of \mathcal{F}_n with respect to an Abhyankar valuation	12
5. Monomial valuation case: growth of the denominators	19
6. Approximation of monomial valuations by divisorial monomial valuations	23
7. A generalization of Abhyankar-Jung Theorem	30
8. Diophantine Approximation	39
Notations	40
References	41

1. INTRODUCTION

When \mathbb{k} is an algebraically closed field of characteristic zero, we can always express the roots of a polynomial with coefficients in the field of power series over \mathbb{k} , denoted by $\mathbb{k}((t))$, as formal Laurent series in $t^{\frac{1}{k}}$ for some positive integer k . This result was known by Newton (at least formally see [BK] p. 372) and had been rediscovered by Puiseux in the complex analytic case [Pu1], [Pu2] (see [BK] or [Cu] for a presentation of this result). A modern way to reformulate this fact is to say that an algebraic closure of $\mathbb{k}((t))$ is the field of Puiseux power series \mathbb{P} defined in the following way:

$$\mathbb{P} := \bigcup_{k \in \mathbb{N}} \mathbb{k} \left(\left(t^{\frac{1}{k}} \right) \right).$$

2000 *Mathematics Subject Classification.* Primary: 12J20. Secondary: 11J25, 12F99, 13J05, 14B05, 32B10.

The proof of this result, called the Newton-Puiseux method, consists essentially in constructing the roots of a polynomial $P(Z) \in \mathbb{k}[[t]][Z]$ by successive approximations in a similar way to Newton method in numerical analysis. These approximations converge since $\mathbb{k}((t^{\frac{1}{k}}))$ is a complete field with respect to the Krull topology.

This result, applied to a polynomial with coefficients in $\mathbb{k}[[t]]$ defining a germ of algebroid plane curve $(X, 0)$, provides an uniformization of this germ, i.e. a parametrization of this germ.

On the other hand this description of the algebraic closure of $\mathbb{k}((t))$ describes very easily the Galois group of $\mathbb{k}((t)) \rightarrow \mathbb{P}$, since this one is generated by the multiplication of the k -th roots of unity with $t^{\frac{1}{k}}$ for any positive integer k . In particular the conjugates of any convergent power series in $\mathbb{C}\{t^{\frac{1}{k}}\}$ are also in $\mathbb{C}\{t^{\frac{1}{k}}\}$.

When \mathbb{k} is a characteristic zero field (not necessarily algebraically closed), we can prove in the same way that an algebraic closure of $\mathbb{k}((t))$ is

$$(1) \quad \mathbb{P} := \bigcup_{\mathbb{k}'} \bigcup_{k \in \mathbb{N}} \mathbb{k}' \left(\left(t^{\frac{1}{k}} \right) \right).$$

where the first union runs over all finite field extensions $\mathbb{k} \rightarrow \mathbb{k}'$.

It is tempting to find such a similar result for the algebraic closure of the field of power series in n variables, $\mathbb{k}((x_1, \dots, x_n))$, for $n \geq 2$. But it appears easily that the algebraic closure of this field admits a really more complicated description and considering only power series depending on $x_1^{\frac{1}{k}}, \dots, x_n^{\frac{1}{k}}$ is not sufficient. For instance it is easy to see that a root square of $x_1 + x_2$ can not be expressed as such a power series.

Nevertheless there exist positive results in some specific cases, the more famous one being the Abhyankar-Jung theorem:

Theorem (Abhyankar-Jung Theorem). *If \mathbb{k} is an algebraically closed field of characteristic zero, then any polynomial with coefficients in $\mathbb{k}[[x_1, \dots, x_n]]$, whose discriminant has the form $ux_1^{\alpha_1} \dots x_n^{\alpha_n}$ where $u \in \mathbb{k}[[x_1, \dots, x_n]]$ is a unit and $\alpha_1, \dots, \alpha_n \in \mathbb{Z}_{\geq 0}$, has its roots in $\mathbb{k}[[x_1^{\frac{1}{k}}, \dots, x_n^{\frac{1}{k}}]]$ for some positive integer k .*

Such a polynomial is called a quasi-ordinary polynomial and this theorem asserts that the roots of quasi-ordinary polynomials are Puiseux power series in several variables. This result has first being proven by Jung in the complex analytic case, then by Abhyankar in the general case ([Ju], [Ab]).

The understanding of the algebraic closure of $\mathbb{k}((x_1, \dots, x_n))$ is motivated by the fact that it would provide an uniformization of any germ of algebroid hypersurface defined over \mathbb{k} . In the general case, a naive approach involves the use of Newton-Puiseux theorem n times (i.e. the formula (1) for the algebraic closure of $\mathbb{k}((t))$). For example in the case $n = 2$, this means that the algebraic closure of $\mathbb{k}((x_1, x_2))$ is included in

$$\mathbb{L} := \bigcup_{k_2 \in \mathbb{N}} \bigcup_{k_1 \in \mathbb{N}} \mathbb{k} \left(\left(x_1^{\frac{1}{k_1}} \right) \right) \left(\left(x_2^{\frac{1}{k_2}} \right) \right).$$

But this field, which is algebraically closed, is very much larger than the algebraic closure of $\mathbb{k}((x_1, x_2))$ (see [Sa] for some thoughts about this). Moreover the action of the k_1 -th and k_2 -th roots of unity are not sufficient to generate the Galois group of the algebraic closure since there exist elements of $\mathbb{k}((x_1))(x_2)$ which are algebraic over $\mathbb{k}((x_1, x_2))$ but are not

in $\mathbb{k}((x_1, x_2))$. For instance consider

$$x_1 \sqrt{1 + \frac{x_2}{x_1}} = \sum_{i \in \mathbb{N}} a_i \frac{1}{x_1^{i-1}} x_2^i \in \mathbb{Q}((x_1))((x_2)) \setminus \mathbb{Q}((x_1, x_2))$$

for some well chosen rational numbers $a_i \in \mathbb{Q}$, $i \in \mathbb{Z}_{\geq 0}$.

Nevertheless a deeper analysis of the Newton-Puiseux method leads to the fact that it is enough to consider the field of fraction of the ring of elements

$$f = \sum_{(l_1, l_2) \in \mathbb{Z}^2} a_{l_1, l_2} x_1^{\frac{l_1}{k_1}} x_2^{\frac{l_2}{k_2}} \in \mathbb{L}$$

for some $k_1, k_2 \in \mathbb{N}$ whose support (i.e. $\text{Supp}(f) := \{(l_1, l_2) \in \mathbb{Z}^2 / a_{l_1, l_2} \neq 0\}$) is included in a rational strictly convex cone of \mathbb{R}^2 (this result has been proven by MacDonald [McD] - see also [Go], [Aro], [AI], [SV]). But once more, for any rational strictly convex cone of \mathbb{R}^2 , denoted by σ , $\mathbb{R}_{\geq 0}^2 \subsetneq \sigma$, there exist elements whose support is in σ but that are not algebraic over $\mathbb{k}((x_1, x_2))$.

One of the main difficulties comes from the fact that $\mathbb{k}((x_1, \dots, x_n))$ is not a complete field with respect to the topology induced by the maximal ideal of $\mathbb{k}[[x_1, \dots, x_n]]$ (called the Krull topology; it is induced by the following norm $\left| \frac{f}{g} \right| := e^{\text{ord}(g) - \text{ord}(f)}$ for any $f, g \in \mathbb{k}[[x_1, \dots, x_n]]$, $g \neq 0$, where $\text{ord}(f)$ is the order of the series f in the usual sense). Indeed, in order to apply the Newton-Puiseux method we have to work with a complete field (or a complete local ring) since the roots are constructed by successive approximations. A very natural idea is to replace $\mathbb{k}((x_1, \dots, x_n))$ by its completion. But choosing the Krull topology is arbitrary and we may choose any other norm (or valuation) on this field. On the other hand the completion of $\mathbb{k}((x_1, \dots, x_n))$ is not algebraic over $\mathbb{k}((x_1, \dots, x_n))$, thus the fields we construct in this way are bigger than the algebraic closure of $\mathbb{k}((x_1, \dots, x_n))$. In fact we need to replace the completion of $\mathbb{k}((x_1, \dots, x_n))$ by its henselization in the completion. The problem is that there is no criterion for distinguish elements of the henselization from others elements of the completion.

The aim of this work is to investigate the use of the Newton-Puiseux method with respect to "tame" valuations (i.e. replace $\mathbb{k}((x_1, \dots, x_n))$ by its completion for this valuation). By a "tame" valuation we mean a rank one (or real valued) valuation that satisfies the equality in the Abhyankar inequality (see Definition 2.1). These valuations are called Abhyankar valuations (cf. [ELS]) or quasi-monomial valuations (cf. [FJ]) and, essentially, these are monomial valuations after some sequence of blowing-ups. This is the first part of this work. If ν is such a valuation, we denote by $\widehat{\mathbb{K}}_\nu$ the completion of $\mathbb{k}((x_1, \dots, x_n))$ for the topology induced by this valuation. This field will takes the role of $\mathbb{k}((t))$ in the classical Newton-Puiseux method. Then we have to define the elements that will take the role of $t^{\frac{1}{k}}$. This is where the first difficulty appears, since instead of working over $\widehat{\mathbb{K}}_\nu$, we need to work over the graded ring associated to ν . Both are isomorphic but there is no canonical isomorphism between them. In the case of $\mathbb{k}((t))$, such an isomorphism is defined by identifying the \mathbb{k} -vector space of homogeneous elements of degree i of the graded ring with the \mathbb{k} -vectors space of homogeneous polynomials of degree i , i.e. $\mathbb{k} \cdot t^i$. But this identification depends on the choice of a uniformizer of $\mathbb{k}[[t]]$. In the case of $\mathbb{k}((x_1, \dots, x_n))$ an isomorphism will be determined by the choice of "coordinates" such that the valuation ν is monomial in these coordinates (cf. Remark 3.1). Nevertheless when such an isomorphism is chosen, we are able to define the elements that will take the role of $t^{\frac{1}{k}}$, this the aim of Section 3. These elements are called

homogeneous elements with respect to ν (cf. Definitions 3.12 and 3.14). These are defined as being the roots of quasi-homogeneous polynomials with coefficients in the graded ring of $\mathbb{k}[[x_1, \dots, x_n]]$ for the valuation ν . If \mathbb{k} is the field of complex numbers and the weights of the monomial valuations are positive integers, we can see these homogeneous elements as weighted homogeneous algebraic (multivalued) functions. In fact we can replace $\widehat{\mathbb{K}}_\nu$ by a smaller field, the subfield of $\widehat{\mathbb{K}}_\nu$ whose elements have support included in a finitely generated semi-group of $\mathbb{R}_{\geq 0}$. Let us remark that this field is similar to the field of generalized power series $\cup_{\Gamma} \mathbb{C}((t^\Gamma))$ where the sum runs over all finitely generated semigroups Γ of $\mathbb{R}_{\geq 0}$ (see [Ri] for instance). Our first result is that the inductive limit of the extensions of $\widehat{\mathbb{K}}_\nu$ by homogeneous elements with respect to ν is algebraically closed (see Theorem 4.2). This field is $\lim_{\substack{\longrightarrow \\ \gamma_1, \dots, \gamma_s}} \widehat{\mathbb{K}}_\nu[\gamma_1, \dots, \gamma_s]$ where the limit runs over all subsets $\{\gamma_1, \dots, \gamma_s\}$ of homogeneous

elements with respect to ν and is denoted by $\overline{\mathbb{K}}_\nu$. The field extension $\mathbb{k}((x_1, \dots, x_n)) \rightarrow \overline{\mathbb{K}}_\nu$ factors through the field extension $\mathbb{k}((x_1, \dots, x_n)) \rightarrow \widehat{\mathbb{K}}_\nu$. While the Galois group of the field extension $\widehat{\mathbb{K}}_\nu \rightarrow \overline{\mathbb{K}}_\nu$ is easily described by the Galois group of weighted homogeneous polynomials, the Galois group of the algebraic closure of $\mathbb{k}((x_1, \dots, x_n))$ in $\widehat{\mathbb{K}}_\nu$ is more complicated. So it is very natural to study irreducible polynomials over $\mathbb{k}((x_1, \dots, x_n))$ which remain irreducible over $\widehat{\mathbb{K}}_\nu$, since their Galois group will be described by the Galois group of weighted homogeneous polynomials. Proposition 4.11 shows that this property is an open property with respect to the Krull topology.

Then we investigate more deeply the particular case of monomial valuations. In Section 5, using an idea of Tougeron [To] based on a work of Gabrielov [Ga], for any monomial valuation ν we construct a field, smaller than the ones constructed previously using the Newton-Puiseux method, and containing an algebraic closure of $\mathbb{k}((x_1, \dots, x_n))$. The tool we use here is an effective version of the Implicit Function Theorem (see Proposition 5.5). The elements we need to consider are of the form

$$(2) \quad \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}}$$

where a_i and δ are weighted homogeneous polynomial for the weights corresponding to the monomial valuation considered, Λ is a finitely sub-semigroup of $\mathbb{R}_{\geq 0}$, $\nu\left(\frac{a_i}{\delta^{m(i)}}\right) = i$ for all $i \in \Lambda$ and $i \mapsto m(i)$ is bounded by an affine function. In the case the weights are \mathbb{Q} -linearly independent we recover the result of MacDonald (see Theorem 6.9).

In Section 7 we make a topological and complex analytical study of polynomials with coefficients in $\mathbb{C}\{x_1, \dots, x_n\}$ whose discriminant is weighted homogeneous. This study has been inspired by the work of Tougeron in [To] and more particularly by Remarque 2.7 of [To] where it is noticed that elements of the form (2) define analytic functions on an open domain of \mathbb{C}^n which is the complement of some hornshaped neighborhood of $\{\delta = 0\}$ (see Definition 7.1). This study is possible in the case of monomial valuations whose weights are positive integers. To obtain the same results in the case of general monomial valuations we need to approximate general monomial valuations by divisorial monomial valuations, i.e. monomial valuations whose weights are positive integers. This is the subject of Section 6.

One of the results we obtain in Section 7 is the following generalization of Abhyankar-Jung Theorem for polynomials whose discriminant is weighted homogeneous :

Theorem. 7.8 *Let \mathbb{k} be a field of characteristic zero, $\alpha \in \mathbb{R}_{>0}^n$ and let $P(Z) \in \mathbb{k}[\mathbf{x}][Z]$ be a monic polynomial such that its discriminant is equal to δu where $\delta \in \mathbb{k}[\mathbf{x}]$ is a weighted homogeneous polynomial for the weights $\alpha_1, \dots, \alpha_n$ and $u \in \mathbb{k}[\mathbf{x}]$ is a unit. Let us set*

$N := \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n)$. Then there exist $\gamma_1, \dots, \gamma_N$ integral homogeneous elements with respect to ν_α and a weighted homogeneous polynomial for the weights $\alpha_1, \dots, \alpha_n$ denoted by $c(\mathbf{x}) \in \mathbb{k}[\mathbf{x}]$ such that the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\mathbb{k}'[[\mathbf{x}]][\gamma_1, \dots, \gamma_N]$ where $\mathbb{k} \rightarrow \mathbb{k}'$ is a finite field extension.

Indeed, in the case $N = n$, i.e. $\alpha_1, \dots, \alpha_n$ are \mathbb{Q} -linearly independent, the only weighted homogeneous polynomials are the monomials and the integral homogeneous elements with respect to ν_α are of the form \mathbf{x}^β where $\beta \in \mathbb{Q}_{\geq 0}^n$ (see Remark 3.15). Abhyankar-Jung Theorem simply asserts that we may choose $c(\mathbf{x}) = 1$ in this case (see Corollary 7.9). This result (along with Theorem 7.5) shows that the Galois group of a monic polynomial with coefficients in $\mathbb{k}[[x_1, \dots, x_n]]$ whose discriminant is weighted homogeneous is isomorphic to the Galois groups of one weighted homogeneous polynomial (see Remark 7.6).

Finally in Section 8 we give a result of Diophantine approximation (it is just an easy generalization of [Ro1] and [II]) that gives a necessary condition for an element of $\widehat{\mathbb{K}}_\nu$ to be algebraic over $\mathbb{k}((x_1, \dots, x_n))$.

At the end we give a list of notations for the convenience of the reader.

Let us mention that this work has been motivated by the understanding of the paper [To] of Tougeron where the study we make for monomial valuations is made in the case of the (x_1, \dots, x_n) -adic valuation of $\mathbb{k}((x_1, \dots, x_n))$.

I would like to thank Guy Casale and Adam Parusiński for their answers to my questions regarding the proofs of Lemma 7.4 and Lemma 7.2 respectively.

2. NOTATIONS

Let \mathbb{N} denote the set of positive integers and $\mathbb{Z}_{\geq 0}$ the set of non-negative integers. Let \mathbf{x} denote the multi-variable (x_1, \dots, x_n) where $n \geq 2$. We set $\mathcal{F}_n := \mathbb{k}[[\mathbf{x}]] = \mathbb{k}[[x_1, \dots, x_n]]$ where \mathbb{k} is a field of characteristic zero and we denote by \mathbb{K}_n its fraction field and by \mathfrak{m} its maximal ideal. A valuation on \mathcal{F}_n is a function $\nu : \mathcal{F}_n^* \rightarrow \Gamma^+$, where Γ is an ordered subgroup of \mathbb{R} and $\Gamma^+ := \Gamma \cap \mathbb{R}_{\geq 0}$, such that $\nu(fg) = \nu(f) + \nu(g)$ and $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$ for any $f, g \in \mathcal{F}_n$. We will also impose that $\nu(\mathbb{k}^*) = 0$ and $\nu(0) = \infty$ where $\infty > i$ for any $i \in \Gamma$. The valuation ν extends to \mathbb{K}_n by $\nu\left(\frac{f}{g}\right) := \nu(f) - \nu(g)$ for any $f, g \in \mathcal{F}_n, g \neq 0$. We will always assume that $\nu : \mathbb{K}_n \rightarrow \Gamma$ is surjective. In this case Γ is called the *value group* of ν . Let us denote by V_ν the valuation ring of ν :

$$V_\nu := \left\{ \frac{f}{g} \mid f, g \in \mathcal{F}_n, \nu(f) \geq \nu(g) \right\}.$$

Let us denote by \widehat{V}_ν its completion. Then \widehat{V}_ν is an equicharacteristic complete valuation ring and its residue field is $\mathbb{k}_\nu := \frac{V_\nu}{\mathfrak{m}V_\nu}$.

In this paper we will only consider a particular case of valuations, called Abhyankar valuations:

Definition 2.1. A valuation ν is called an *Abhyankar valuation* if the following properties hold:

- i) For any $f \in \mathcal{F}_n$, $\nu(f) > 0$ if and only if $f \in \mathfrak{m}$.
- ii) $\text{tr.deg}_{\mathbb{k}} \mathbb{k}_\nu + \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q} = n$ (Abhyankar's Equality).

Remark 2.2. If $\dim_{\mathbb{Q}} \Gamma \otimes \mathbb{Q} = 1$, then $\Gamma \simeq \mathbb{Z}$. Otherwise Γ is a dense subgroup of \mathbb{R} .

Example 2.3. The first example is the \mathfrak{m} -adic valuation denoted by ord and defined by

$$\text{ord}(f) := \max\{n \in \mathbb{N} \mid f \in \mathfrak{m}^n\} \quad \forall f \in \mathcal{F}_n \setminus \{0\}.$$

In this case its value group Γ is equal to \mathbb{Z} .

Example 2.4. Let $\alpha := (\alpha_1, \dots, \alpha_n) \in (\mathbb{R}_{>0})^n$. Let us denote by ν_α the monomial valuation defined by $\nu_\alpha(x_i) := \alpha_i$ for $1 \leq i \leq n$. For instance $\nu_{(1, \dots, 1)} = \text{ord}$.

Here we have $\Gamma = \mathbb{Z}\alpha_1 \oplus \dots \oplus \mathbb{Z}\alpha_n$.

Example 2.5. If Γ is isomorphic to \mathbb{Z} and ν is an Abhyankar valuation, then ν is a *divisorial valuation*. For such valuation there exists a proper birational dominant map $\pi : X \rightarrow \text{Spec}(\mathcal{F}_n)$ and E an irreducible component of exceptional locus of π such that ν is the composition of π^* with the \mathfrak{m}_E -adic valuation of the ring $\mathcal{O}_{X,E}$.

Remark 2.6. By Proposition 2.8 [ELS], for any Abhyankar valuation ν there exists a proper birational dominant map $\pi : X \rightarrow \text{Spec}(\mathcal{F}_n)$, a point p in the exceptional locus of π (not necessarily closed) and a regular system of parameters z_1, \dots, z_r of the local ring of X at p and $\alpha \in \mathbb{R}_{>0}^r$, such that ν is the composition of π_* with the monomial valuation μ_α at p in the coordinates z_1, \dots, z_r of weights $\alpha_1, \dots, \alpha_r$. In the case $n = 2$ p is a closed point (see Proposition 6.38 [FJ]). In any case the completion of $\mathcal{O}_{X,p}$ is a ring of formal power series over the residue field of ν on X : $\widehat{\mathcal{O}}_{X,p} \simeq \mathbb{L}[[z_1, \dots, z_r]]$ where $\mathbb{k} \rightarrow \mathbb{L}$ is a field extension of transcendence degree $n - r$ (here $\mathbb{L} = \frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p}$). Since π is birational then $\mathbb{K}_n = \text{Frac}(\mathcal{O}_{X,p})$ and since $\mu_\alpha \circ \varphi = \nu$ then π_* induces an isomorphism $\varphi : V_\nu \rightarrow V_{\mu_\alpha}$. Thus π^* induces an isomorphism $\widehat{V}_\nu \rightarrow \widehat{V}_{\mu_\alpha}$.

Remark 2.7. If $n = 2$, in fact any discrete valuation (i.e. $\Gamma = \mathbb{Z}$) satisfying Property i) of Definition 2.1 is an Abhyankar valuation [HOV].

Example 2.8. Let ν_α be a monomial valuation as before. Any power series $g \in \mathcal{F}_n$ can be written $g = \sum_i g_i$ where g_i is a weighted homogeneous polynomial of degree $i \in \Gamma^+$ if x_j has weight α_j . Moreover the set of indices $i \in \Gamma^+$ such that $g_i \neq 0$ is a well-ordered set. Let us denote by i_0 the least $i \in \Gamma^+$ such that $g_i \neq 0$. Then we can write formally

$$g = g_{i_0} \left(1 + \sum_{i > i_0} \frac{g_i}{g_{i_0}} \right)$$

and this equality is satisfied in \widehat{V}_{ν_α} . Now if $f \in \mathcal{F}_n$, $g \neq 0$ and $\nu(f) \geq \nu(g)$ we can write

$$\frac{f}{g} = \left(\sum_i \frac{f_i}{g_{i_0}} \right) \left(1 + \sum_{i > i_0} \frac{g_i}{g_{i_0}} \right)^{-1}$$

where $f = \sum_i f_i$ with f_i is a weighted homogeneous polynomial of degree $i \in \Gamma^+$ if x_j has weight α_j .

Thus any element of V_{ν_α} is of the form $\sum_{i \geq 0} \frac{a_i(\mathbf{x})}{b_i(\mathbf{x})}$ where $a_i(\mathbf{x})$ and $b_i(\mathbf{x})$ are quasi-homogeneous polynomials if x_j has weight α_j and $\nu_\alpha \left(\frac{a_i(\mathbf{x})}{b_i(\mathbf{x})} \right) = i$ for any $i \in \mathbb{R}$. Moreover the set of elements of this form is exactly \widehat{V}_{ν_α} .

Let us denote by $\widehat{\mathbb{K}}_\nu$ the fraction field of \widehat{V}_ν . The valuation ν defines an ultrametric norm on $\widehat{\mathbb{K}}_\nu$, denoted by $|\cdot|_\nu$, defined by

$$\left| \frac{f}{g} \right|_\nu = e^{\nu(g) - \nu(f)} \quad \forall f \in \mathcal{F}_n, g \in \mathcal{F}_n \setminus \{0\}.$$

Then $\widehat{\mathbb{K}}_\nu$ is the completion of \mathbb{K}_n for the topology induced by this norm and this norm (thus the valuation ν) extends canonically on $\widehat{\mathbb{K}}_\nu$. We shall also denote this extension by ν .

Let us denote by $\mathbb{K}_\nu^{\text{alg}}$ the algebraic closure of \mathbb{K}_n in $\widehat{\mathbb{K}}_\nu$. We also denote by V_ν^{alg} the ring of elements of \widehat{V}_ν which are algebraic over \mathbb{K}_n : $V_\nu^{\text{alg}} := \mathbb{K}_\nu^{\text{alg}} \cap \widehat{V}_\nu$. We have the following lemma:

Lemma 2.9. *The ring V_ν^{alg} is a valuation ring (associated to the valuation ν) and $\mathbb{K}_\nu^{\text{alg}}$ is its fraction field. Moreover $V_\nu \longrightarrow V_\nu^{\text{alg}}$ is the henselization of V_ν in \widehat{V}_ν .*

Proof. If $f, g \in V_\nu^{\text{alg}}$ and $\nu(f) \geq \nu(g)$, then $\frac{f}{g} \in \mathbb{K}_\nu^{\text{alg}} \cap \widehat{V}_\nu$. If $f \in \mathbb{K}_\nu^{\text{alg}}$, then there exists $N \in \mathbb{N}$ such that $x_1^N f \in \mathbb{K}_\nu^{\text{alg}} \cap \widehat{V}_\nu = V_\nu^{\text{alg}}$ since $\nu(x_1^N) > 0$.

By construction the elements of the henselization of V_ν are algebraic over V_ν . On the other any element of \widehat{V}_ν which is algebraic over V_ν is in the Henselization of V_ν (see Corollary 1.2.1 [M-B]). \square

Thus we can summarize the situation with the following commutative diagram, where the bottom part corresponds to the quotient fields of rings of the upper part:

$$\begin{array}{ccccc} \mathcal{F}_n & \longrightarrow & V_\nu & \longrightarrow & \widehat{V}_\nu \\ & & \downarrow & \searrow & \downarrow \\ & & \mathbb{K}_n & \xrightarrow{\quad} & \mathbb{K}_\nu^{\text{alg}} \\ & & \downarrow & \nearrow & \downarrow \\ & & \mathbb{K}_\nu^{\text{alg}} & \xrightarrow{\quad} & \widehat{\mathbb{K}}_\nu \end{array}$$

3. HOMOGENEOUS ELEMENTS WITH RESPECT TO AN ABHYANKAR VALUATION

Let A be an integral domain and let $\nu : A \longrightarrow \Gamma^+$ be a valuation where Γ is a subgroup of \mathbb{R} . We have $\text{Gr}_\nu A = \bigoplus_{i \in \Gamma^+} \frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}_{\nu,i}^+}$ where $\mathfrak{p}_{\nu,i} := \{f \in A / \nu(f) \geq i\}$ and $\mathfrak{p}_{\nu,i}^+ := \{f \in A / \nu(f) > i\}$.

Remark 3.1. Let us consider a monomial valuation ν , let us say $\nu := \nu_\alpha$ where $\alpha \in \mathbb{R}_{>0}^n$. In this case $\frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}_{\nu,i}^+}$ is isomorphic to the \mathbb{k} -vector space of rational fractions $\frac{a(\mathbf{x})}{b(\mathbf{x})}$ where $a(\mathbf{x})$ and $b(\mathbf{x})$ are quasi-homogeneous polynomials if x_j has weight α_j and $\nu_\alpha \left(\frac{a(\mathbf{x})}{b(\mathbf{x})} \right) = i$. Thus, by Example 2.8 $\widehat{\text{Gr}_\nu V_\nu}$ and \widehat{V}_ν are \mathbb{k} -isomorphic.

Let us now consider a general Abhyankar valuation ν . By Remark 2.6, there exists a proper birational dominant map $\pi : X \longrightarrow \text{Spec}(\mathcal{F}_n)$, a point p in the exceptional locus of π and a regular system of parameter of $\mathcal{O}_{X,p}$, z_1, \dots, z_r , such that ν is the composition of π_* with a monomial valuation μ at p in the coordinates z_1, \dots, z_r . The valuation μ is defined on $\mathcal{O}_{X,p}$ and let us denote by $\widehat{\mu}$ the corresponding monomial valuation defined on $\widehat{\mathcal{O}}_{X,p} \simeq \mathbb{L}[[z_1, \dots, z_r]]$. By Remark 2.6, π induces an isomorphism $\widehat{V}_\nu \simeq \widehat{V}_\mu$. Moreover $\widehat{V}_\mu = \widehat{V}_{\widehat{\mu}}$

and $\text{Gr}_\nu V_\nu = \text{Gr}_\mu \mathcal{O}_{X,p} = \text{Gr}_\mu \widehat{\mathcal{O}_{X,p}}$. Thus $\widehat{\text{Gr}_\nu V_\nu}$ and \widehat{V}_ν are \mathbb{k} -isomorphic by the monomial case.

Thus we see that the choice of a proper birational map π as in Remark 2.6 gives an isomorphism between $\widehat{\text{Gr}_\nu V_\nu}$ and \widehat{V}_ν . A different choice of π and z_1, \dots, z_r gives an other isomorphism between these two rings.

Remark 3.2. The ring $\widehat{\text{Gr}_\nu V_\nu}$ is isomorphic to the ring of generalized power series $\mathbb{k}_\nu[[t^{\Gamma^+}]]$ where t is a single variable.

Remark 3.3. The elements of $\widehat{\text{Gr}_\nu V_\nu}$ are the elements of the form $\sum_{i \in \Lambda} a_i$ where $a_i \in \frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}_{\nu,i}^+}$ for all $i \in \Lambda$ where Λ is either a finite set, either a countable subset of $\mathbb{R}_{\geq 0}$ with no accumulation point.

Definition 3.4. Let Γ^+ be a sub-semigroup of $\mathbb{R}_{\geq 0}$. A Γ^+ -graded ring is a ring A that has a direct sum decomposition, $A = \bigoplus_{i \in \Gamma^+} A_i$, such that $A_i A_j \subset A_{i+j}$ for any $i, j \in \Gamma^+$.

The completion of A is the set of elements that are written as a series $\sum_{i \in \Lambda} a_i$ where $\Lambda \subset \Gamma^+$ is either a finite set, either a countable subset of $\mathbb{R}_{> 0}$ with no accumulation point, and $a_i \in A_i$ for any $i \in \Lambda$. The completion of A is denoted by \widehat{A} or $\widehat{\bigoplus_{i \in \Gamma^+} A_i}$.

A complete (Γ^+) -graded ring is the completion of a (Γ^+) -graded ring.

Remark 3.5. Let A be a complete graded ring. If A_0 is a field then A is a local ring and its maximal ideal is $\mathfrak{m} := \bigoplus_{i > 0} A_i$.

For any $a \in A$ we can write $a = \sum_{i \in \Gamma^+} a_i$ where $a_i \in A_i$ for any i . If $a \neq 0$ let us set $\nu(a) := \inf\{i \in \Gamma^+ \mid a_i \neq 0\}$. Set $\nu(0) = \infty$. Then ν is an order function, i.e. $\nu(ab) \geq \nu(a) + \nu(b)$ and $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$. Moreover ν is a valuation if and only if A is an integral domain. The order function ν is called the order function of A .

Example 3.6. The rings $\text{Gr}_\nu \mathcal{F}_n$ and $\text{Gr}_\nu V_\nu$ are Γ^+ -graded rings and $\widehat{\text{Gr}_\nu \mathcal{F}_n}$ and $\widehat{\text{Gr}_\nu V_\nu}$ are complete Γ^+ -graded rings.

Definition 3.7. Let $A = \widehat{\bigoplus_{i \in \Gamma^+} A_i}$ be a complete Γ^+ -graded ring. Let $a \in A$, $a = \sum_{i \in \Gamma^+} a_i$, $a_i \in A_i$ for any i . The support of a is the subset I of Γ^+ defined by $i \in I$ if and only if $a_i \neq 0$. We denote this set I by $\text{Supp}(a)$.

Definition 3.8. Let us fix a \mathbb{k} -isomorphism φ between $\widehat{\text{Gr}_\nu V_\nu}$ and \widehat{V}_ν as in Remark 3.1. Let $a \in \widehat{V}_\nu$ and let us write $\varphi(a) = \sum_{i \in \Gamma^+} a_i$ with $a_i \in \frac{\mathfrak{p}_{\nu,i}}{\mathfrak{p}_{\nu,i}^+}$. The ν -support with respect to φ of a is the subset of Γ^+ defined as

$$\text{Supp}_{\nu, \varphi}(a) := \{i \in \Gamma^+ \mid a_i \neq 0\}.$$

When the isomorphism is clear from the context we will skip the mention of φ and denote the ν -support of a by $\text{Supp}_\nu(a)$.

Proposition 3.9. Let ν be an Abhyankar valuation and φ a \mathbb{k} -isomorphism between $\widehat{\text{Gr}_\nu V_\nu}$ and \widehat{V}_ν as in Remark 3.1. Then there exists a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$, denoted by Λ , such that the ν -support of any element of \mathcal{F}_n with respect to φ is included in Λ .

Proof. By Remark 2.6, we may assume that ν is a monomial valuation. Thus the proposition comes from the following lemma: \square

Lemma 3.10. Let Σ be a strictly convex rational cone of \mathbb{R}^n . Let $\alpha \in \mathbb{R}_{> 0}^n$ such that $\langle \alpha, \beta \rangle > 0$ for any $\beta \in \Sigma$, $\beta \neq 0$. Then there exists a finitely generated subgroup of $\mathbb{R}_{\geq 0}$, denoted by Λ , such that $\text{Supp}_{\nu_\alpha}(f) \subset \Lambda$ for any $f \in \mathbb{k}[[x^\beta, \beta \in \Sigma \cap \mathbb{Z}^n]]$.

Proof. By Gordan Lemma, $\Sigma \cap \mathbb{Z}^n$ is a finitely generated semigroup, let say $\Sigma \cap \mathbb{Z}^n$ is generated by u_1, \dots, u_k . Let us denote $r_i := \langle \alpha, u_i \rangle$, $1 \leq i \leq k$. Since any element of $\Sigma \cap \mathbb{Z}^n$ is a \mathbb{N} -linear combination of u_1, \dots, u_k , then $\langle \alpha, \beta \rangle$ is a \mathbb{N} -linear combination of r_1, \dots, r_k for any $\beta \in \Sigma \cap \mathbb{Z}^n$. Let us denote by Λ the semigroup of $\mathbb{R}_{\geq 0}$ generated by r_1, \dots, r_k . Then $\text{Supp}_{\nu_\alpha}(f) \subset \Lambda$. \square

From now on we will fix a \mathbb{k} -isomorphism φ between $\widehat{\text{Gr}_\nu V_\nu}$ and \widehat{V}_ν induced by a proper birational map π as in Remark 3.1 and we will skip the mention of it in the following. There are several reasons for that. The first one is that we are interested by effective results on the algebraic elements over \mathcal{F}_n , thus we are interested by valuations which are given effectively and this will be the case essentially through a map π as in Remark 2.6. In particular we will investigate more deeply the case of monomial valuations and, in this case, the set of variables x_1, \dots, x_n will be fixed from the beginning, thus φ is quite natural in this case. The last reason is that we will give properties on the ν -support of algebraic elements, and Proposition 3.9 will allows us to consider only elements whose ν -support is included in a finitely generated sub-semigroup of $\mathbb{R}_{>0}$, and this fact does not depends on φ .

Definition 3.11. We will denote by V_ν^{fg} the subset of \widehat{V}_ν of elements whose ν -support is included in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ (when we identify \widehat{V}_ν and $\widehat{\text{Gr}_\nu V_\nu}$ via φ). It is straightforward to check that V_ν^{fg} is a valuation ring. We denote by $\mathbb{K}_\nu^{\text{fg}}$ its fraction field.

Definition 3.12. Let A be a complete Γ -graded domain and let ν be its order function (which is a valuation). Let $d \in \frac{1}{q!}\Gamma$. A *homogeneous element with respect to ν* , is an element γ of a finite extension of A , such that its minimal polynomial $Q(Z)$ is irreducible in $A[Z]$ and has the following form:

$$Z^q + g_1 Z^{q-1} + \dots + g_q$$

where $g_k \in A_{dk}$ for $1 \leq k \leq q$. In this case the degree of γ is d .

Example 3.13. Let $\alpha \in \mathbb{R}_{>0}^n$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$. Then

$$\Gamma = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_n$$

and for any $i \in \Gamma$ there exists a unique $(\beta_{i,1}, \dots, \beta_{i,n}) \in \mathbb{Z}^n$ such that

$$i = \beta_{i,1}\alpha_1 + \dots + \beta_{i,n}\alpha_n.$$

Thus if $i \in \Gamma^+$ this means that $\frac{\mathbf{p}_{\nu_\alpha, i}}{\mathbf{p}_{\nu_\alpha, i}^+}$ is the one dimensional \mathbb{k} -vector space generated by $x_1^{\beta_{i,1}} \dots x_n^{\beta_{i,n}}$. Thus if $g_k \in \frac{\mathbf{p}_{\nu_\alpha, dk}}{\mathbf{p}_{\nu_\alpha, dk}^+}$ for $1 \leq k \leq q$, then

$$Z^q + g_1 Z^{q-1} + \dots + g_q = x_1^{\beta_{qd,1}} \dots x_n^{\beta_{qd,n}} (T^q + g'_1 T^{q-1} + \dots + g'_q)$$

where $Z = x_1^{\beta_{d,1}} \dots x_n^{\beta_{d,n}} T$ and $g'_1, \dots, g'_q \in \mathbb{k}_\nu$. Here $\beta_{qd,j} \in \mathbb{Z}$ for any j but $\beta_{d,j} = \frac{\beta_{qd,j}}{d}$ may not be an integer. Since \mathbb{k}_ν is an algebraic field extension of \mathbb{k} , then the roots of $T^q + g'_1 T^{q-1} + \dots + g'_q$ are algebraic over \mathbb{k} . Thus homogeneous elements with respect to ν_α are of the form $c\mathbf{x}^\beta$ where c is algebraic over \mathbb{k} and $\beta \in \mathbb{Q}^n$ with $\langle \alpha, \beta \rangle := \alpha_1\beta_1 + \dots + \alpha_n\beta_n \geq 0$.

Definition 3.14. Let $A = \widehat{\text{Gr}_\nu V_\nu}$ and γ be a homogeneous element with respect to ν . Let $Q(Z)$ be its minimal polynomial: $Q(Z) = Z^q + g_1 Z^{q-1} + \dots + g_q$ with $g_k \in \frac{\mathbf{p}_{\nu, dk}}{\mathbf{p}_{\nu, dk}^+}$ for $1 \leq k \leq q$. We say that γ is an *integral homogeneous element with respect to ν* if g_k is the image of an element of $\mathcal{F}_n \cap \mathbf{p}_{\nu, dk}$ for all k .

Example 3.15. Let $\alpha \in \mathbb{R}_{>0}^n$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$ and let us keep the notations of Example 3.13. Then γ is an integral homogeneous element with respect to ν_α if $g_k \in \frac{\mathcal{F}_n \cap \mathfrak{p}_{\nu_\alpha, dk}^+}{\mathcal{F}_n \cap \mathfrak{p}_{\nu_\alpha, dk}^+}$ for $1 \leq k \leq q$. This means that $\beta_{qd,j} \in \mathbb{N}$ for all j . Thus integral homogeneous elements with respect to ν_α are of the form $c\mathbf{x}^\beta$ where c is algebraic over \mathbb{k} and $\beta \in \mathbb{Q}_{\geq 0}^n$.

Example 3.16. Let ν be an Abhyankar valuation and let assume that \mathbb{k} is not algebraically closed. Let c be in the algebraic closure of \mathbb{k} , $c \notin \mathbb{k}$. Then c is a root of a polynomial equation with coefficients in \mathbb{k} and since \mathbb{k} is a subfield of \mathbb{k}_ν , this shows that c is an integral homogeneous element of degree 0 with respect to ν .

Remark 3.17. Let ν be an Abhyankar valuation and let γ be a homogeneous element of degree d with respect to ν . Let us denote by $Q(Z)$ its minimal polynomial, say

$$Q(Z) = Z^q + g_1 Z^{q-1} + \dots + g_q$$

where $g_k \in \frac{\mathfrak{p}_{\nu_\alpha, dk}^+}{\mathfrak{p}_{\nu_\alpha, dk}^+}$ for $1 \leq k \leq q$. Each g_k is the image in $\widehat{\text{Gr}_\nu V_\nu}$ of some fraction $\frac{f_k}{h_k}$ where $f_k, h_k \in \mathcal{F}_n$. Let $h := h_1 \dots h_k$, let $\text{in}(h)$ be the image of h in $\widehat{\text{Gr}_\nu V_\nu}$ and set $\gamma' := \text{in}(h)\gamma$. Then γ' is a homogeneous element annihilating $Z^q + g'_1 Z^{q-1} + \dots + g'_q$ where g'_k is the image of $g_k h^k$ in $\widehat{\text{Gr}_\nu V_\nu}$, thus it is an integral homogeneous element with respect to ν .

Definition 3.18. Let ν be an Abhyankar valuation of \mathcal{F}_n , $P(Z_1, \dots, Z_m) \in \widehat{V}_\nu[Z_1, \dots, Z_m]$ and $\mathbf{d} := (d_1, \dots, d_m) \in \mathbb{R}_{>0}^m$. One says that $P(Z_1, \dots, Z_m)$ is (ν, \mathbf{d}) -homogeneous of degree $d \in \mathbb{R}$ if for any monomial $gZ_1^{\alpha_1} \dots Z_m^{\alpha_m}$ of $P(Z)$ one has $g \in \frac{\mathfrak{p}_{\nu, k}^+}{\mathfrak{p}_{\nu, k}^+}$ with $k + \alpha_1 d_1 + \dots + \alpha_m d_m = d$. In particular if $\alpha \in \mathbb{R}_{>0}^n$, we say that $P(\mathbf{x}) \in \mathbb{k}[\mathbf{x}]$ is (α) -homogeneous if is a weighted homogeneous polynomial when the weight of x_i is α_i .

Remark 3.19. Let ν be an Abhyankar valuation of \mathcal{F}_n . Let γ be a homogeneous element of degree d with respect to ν . Let us denote by $P(Z)$ its minimal monic polynomial. Then $P(Z)$ is (ν, d) -homogeneous.

Conversely if $P(Z) \in \widehat{V}_\nu[Z]$ satisfies $P(\gamma) = 0$ and if $P(Z)$ is (ν, d) -homogeneous, then the divisors of P in $\widehat{V}_\nu[Z]$ are also (ν, d) -homogeneous, thus the minimal polynomial of γ is (ν, d) -homogeneous. Hence γ is a homogeneous element of degree d with respect to ν .

Lemma 3.20. Let γ_1 and γ_2 be two homogeneous elements of degree d_1 and d_2 respectively with respect to the valuation ν and let $k \in \mathbb{Z}$. Then

- i) γ_1^k is homogeneous of degree kd_1 ,
- ii) if $e_1 d_1 = e_2 d_2$ with $e_1, e_2 \in \mathbb{N}$, then $\gamma_1^{e_1} + \gamma_2^{e_2}$ is homogeneous of degree $d_1 e_1$,
- iii) $\gamma_1 \gamma_2$ is homogeneous of degree $d_1 d_2$.

Proof. If γ is homogeneous of degree $d \in \mathbb{Q}$, then γ^k , $k \in \mathbb{N}$, is homogeneous of degree kd . Indeed a polynomial having γ^k as a root is $Q(Z) := \text{Res}_X(P(X), Z - X^k)$ where P is the minimal monic polynomial of γ over $\mathbb{k}(\mathbf{x})$. But $P(X)$ is (ν, d) -homogeneous and $Z - X^k$ is (ν, d, kd) -homogeneous. Thus $Q(Z)$ is (ν, d, kd) -homogeneous, hence (ν, kd) -homogeneous since it does not depend on X . This proves that γ^k is homogeneous of degree kd .

In order to show ii) we may assume, from i), that γ_1 and γ_2 are homogeneous of same weight $d = e_1 d_1 = e_2 d_2$. Let us denote by $P_1(Z)$ and $P_2(Z)$ the minimal monic polynomials of γ_1 and γ_2 respectively. Then $Q(Z) := \text{Res}_X(P_1(Z - X), P_2(X))$ is (ν, d, d) -homogeneous, thus (ν, d) -homogeneous since it does not depend on X . Since $Q(\gamma_1 + \gamma_2) = 0$, $\gamma_1 + \gamma_2$ is homogeneous of degree d .

In order to show iii), let us denote by $P_1(X)$ the minimal monic polynomial of γ_1 (this is a (ν, d_1) -homogeneous polynomial) and $P_2(Z)$ the minimal monic polynomial of γ_2 ((ν, d_2) -homogeneous). Let us denote by k the degree of $P_1(Z)$ let us denote $R(X, Y) := X^k P_1(Y/X)$. Then $\gamma_1 \gamma_2$ is a root of $Q(Z) := \text{Res}_X(R(X, Z), P_2(X))$. Moreover $R(X, Z)$ is $(\nu, d_2, d_1 d_2)$ -homogeneous. Thus $Q(Z)$ is $(\nu, d_1 d_2)$ -homogeneous, which proves that $\gamma_1 \gamma_2$ is homogeneous of degree $d_1 d_2$. \square

Lemma 3.21. *Let $P(T, Z) \in \widehat{V}_\nu[T, Z]$ be a (ν, d_1, d_2) -homogeneous polynomial and let γ_1 be a homogeneous element of degree d_1 with respect to ν . If an element γ_2 belonging to a finite extension of $\mathbb{k}(\mathbf{x})$ satisfies $P(\gamma_1, \gamma_2) = 0$, then γ_2 is a homogeneous element of degree d_2 with respect to ν .*

Proof. Let $Q(T) \in \widehat{V}_\nu[T]$ be a (ν, d_1) -homogeneous polynomial such that $Q(\gamma_1) = 0$. Let us denote $R(Z) = \text{Res}_T(P(T, Z), Q(T))$. Then $R(Z)$ is a (ν, d_2) -homogeneous polynomial such that $R(\gamma_2) = 0$. This proves the result. \square

Remark 3.22. Let A be a complete Γ -graded integral domain and let ν be its order valuation. Let $Q(Z)$ be an irreducible polynomial in $A[Z]$ having the following form:

$$Z^q + g_1 Z^{q-1} + \cdots + g_q$$

where $g_k \in A_{dk}$ for $1 \leq k \leq q$ and $d \in \frac{1}{q!}\Gamma^+$. The ring $B := \frac{A[Z]_{m+(Z)}}{(Q(Z))}$ is an integral domain and ν extends to a valuation of this ring by defining $\nu(Z) := d$ and

$$\nu \left(\sum_{i=0}^{q-1} a_i Z^i \right) := \inf_i \{ \nu(a_i) + i \}.$$

Then B is a complete $\frac{1}{q!}\Gamma$ -graded domain, since B is the completion of $\bigoplus_{i \in \Gamma} \bigoplus_{j=0}^{\min\{\lfloor \frac{i}{d} \rfloor, q\}} A_{i-dj} Z^j$.

Definition 3.23. We denote this ring by $A[\gamma]$. By induction we can define $A[\gamma_1, \dots, \gamma_s]$ where γ_{i+1} is a homogeneous element over $A[\gamma_1, \dots, \gamma_i]$ for $1 \leq i < s$. When $A = \widehat{V}_\nu$, V_ν^{alg} or V_ν^{fg} , we denote by $A[\langle \gamma_1, \dots, \gamma_s \rangle]$ the valuation ring associated to the order valuation of $A[\gamma_1, \dots, \gamma_s]$. In this case the elements of $A[\langle \gamma_1, \dots, \gamma_s \rangle]$ are the elements which are finite sum of terms of the form $b \gamma_1^{j_1} \dots \gamma_s^{j_s}$ where $b \in \text{Frac}(A)$ and $\nu(b) \geq -(j_1 \nu(\gamma_1) + \cdots + j_s \nu(\gamma_s))$.

Definition 3.24. If ν is an Abhyankar valuation we denote by $\overline{V}_\nu := \varinjlim_{\gamma_1, \dots, \gamma_s} \widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_s \rangle]$

the direct limit over all subsets $\{\gamma_1, \dots, \gamma_s\}$ of homogeneous elements with respect to ν and by $\overline{\mathbb{K}}_\nu$ its fraction field. By Remark 3.17 we may restrict the limit over the subsets of integral homogeneous elements.

In the same way we define $\overline{V}_\nu^{\text{fg}} := \varinjlim_{\gamma_1, \dots, \gamma_s} V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_s \rangle]$, $\overline{V}_\nu^{\text{alg}} := \varinjlim_{\gamma_1, \dots, \gamma_s} V_\nu^{\text{alg}}[\langle \gamma_1, \dots, \gamma_s \rangle]$,

the limits being taken over all subsets $\{\gamma_1, \dots, \gamma_s\}$ of (integral) homogeneous elements with respect to ν , and we denote by $\overline{\mathbb{K}}_\nu^{\text{fg}}$ and $\overline{\mathbb{K}}_\nu^{\text{alg}}$ their fraction fields.

The following result gives a bound on the number of homogeneous elements we need to consider:

Proposition 3.25. *Let ν be an Abhyankar valuation on \mathcal{F}_n and let Γ denote its value group. Set $N := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$ and let $\gamma_1, \dots, \gamma_s$ be homogeneous elements with respect to ν . Then there exist homogeneous elements $\gamma'_1, \dots, \gamma'_N$ with respect to ν such that $\widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_s \rangle] = \widehat{V}_\nu[\langle \gamma'_1, \dots, \gamma'_N \rangle]$.*

This equality remains true if we replace \widehat{V}_ν by V_ν^{alg} or V_ν^{fg} and if we consider integral homogeneous elements.

Proof. We will prove this proposition by induction on s . Let $\gamma_1, \dots, \gamma_{N+1}$ be non-zero homogeneous elements with respect to ν . Let d_i be the degree of γ_i , for $1 \leq i \leq N+1$. By assumption on N the d_i 's are \mathbb{Q} -linearly dependent. Thus, after a permutation of the g_i 's, there exists an integer r , $s \geq N$, $p_i \in \mathbb{Z}_{\geq 0}$, $q_i \in \mathbb{N}$, $1 \leq i \leq N+1$, such that

$$(3) \quad \frac{p_1}{q_1}d_1 + \dots + \frac{p_s}{q_s}d_s = \frac{p_{s+1}}{q_{s+1}}d_{s+1} + \dots + \frac{p_{N+1}}{q_{N+1}}d_{N+1}.$$

Set $r_i := \frac{p_1 \dots p_{N+1}}{p_i}$ for $1 \leq i \leq N+1$. Let us denote $\gamma'_i := \gamma_i^{\frac{1}{q_i r_i}}$. Then we have

$$\widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_{N+1} \rangle] \subset \widehat{V}_\nu[\langle \gamma'_1, \dots, \gamma'_{N+1} \rangle].$$

By (3) and Lemma 3.20, $\gamma'_1 \dots \gamma'_s$ and $\gamma'_{s+1} \dots \gamma'_{N+1}$ are homogeneous elements of same degree. By the Primitive Element Theorem, there exists $c \in \mathbb{k}$ such that

$$\mathbb{k}(\mathbf{x})[\gamma'_1 \dots \gamma'_s, \gamma'_{s+1} \dots \gamma'_{N+1}] = \mathbb{k}(\mathbf{x})[\gamma'_1 \dots \gamma'_s + c\gamma'_{s+1} \dots \gamma'_{N+1}],$$

and moreover $\gamma := \gamma'_1 \dots \gamma'_s + c\gamma'_{s+1} \dots \gamma'_{N+1}$ is a homogeneous element with respect to ν of same degree as $\gamma'_1 \dots \gamma'_s$ and $\gamma'_{s+1} \dots \gamma'_{N+1}$. Since $\mathbb{k}(\mathbf{x})[\gamma'_1, \dots, \gamma'_s] = \mathbb{k}(\mathbf{x})[\gamma'_1, \dots, \gamma'_{s-1}, \gamma'_1 \dots \gamma'_s]$ and $\mathbb{k}(\mathbf{x})[\gamma'_{s+1}, \dots, \gamma'_{N+1}] = \mathbb{k}(\mathbf{x})[\gamma'_{s+1}, \dots, \gamma'_N, \gamma'_{s+1} \dots \gamma'_{N+1}]$, we have

$$\mathbb{k}(\mathbf{x})[\gamma'_1, \dots, \gamma'_{N+1}] = \mathbb{k}(\mathbf{x})[\gamma'_1, \dots, \gamma'_{s-1}, \gamma'_{s+1}, \dots, \gamma'_N, \gamma].$$

Thus γ'_s is a finite sum of products of elements $a_i(\mathbf{x}) \in \mathbb{k}(\mathbf{x})$ and powers of $\gamma'_1, \dots, \gamma'_{s-1}, \gamma'_{s+1}, \dots, \gamma'_N, \gamma$ and by homogeneity we may assume that $a_i(\mathbf{x})$ are (ν) -homogeneous. Thus

$$\widehat{V}_\nu[\langle \gamma'_1, \dots, \gamma'_{N+1} \rangle] = \widehat{V}_\nu[\langle \gamma'_1, \dots, \gamma'_{s-1}, \gamma'_{s+1}, \dots, \gamma'_N, \gamma \rangle].$$

This equality remains true if we replace \widehat{V}_ν by V_ν^{alg} or V_ν^{fg} . By Remark 3.17, the result remains true for integral homogeneous elements. \square

4. NEWTON METHOD AND ALGEBRAIC CLOSURE OF \mathcal{F}_n WITH RESPECT TO AN ABHYANKAR VALUATION

Lemma 4.1. *Let (A, \mathfrak{m}) be a complete graded local ring. Let B be the subset of A whose elements are the elements of A whose support is included in a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Then B is a Henselian local domain.*

Proof. Let us prove that B is a ring: let b_1 and b_2 be two elements of B whose supports are included in Λ_1 and Λ_2 respectively. Thus we can write $b_i = \sum_{j \in \Lambda_i} b_{i,j}$ where $b_{i,j}$ is a homogeneous element of degree j for any $i = 1, 2$ and $j \in \Lambda_1$ or Λ_2 . Let Λ be the finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ generated by Λ_1 and Λ_2 . Then $\text{Supp}(b_1 + b_2)$ and $\text{Supp}(b_1 b_2)$ are included in Λ . This proves that B is a ring. Since $B \subset A$, B is a domain.

It is clear that $\mathfrak{m} \cap B$ is an ideal of B . If $b \in B \setminus (\mathfrak{m} \cap B)$, then there exists $a \in A$ such that $ab = 1$. Let us write $b = \sum_{i \in \Lambda} b_i$ where b_i is homogeneous of degree i and Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Since $b \notin \mathfrak{m}$, then $b_0 \neq 0$. In this case we have

$$a = b^{-1} = \frac{1}{b_0} \left(1 + \sum_{i \in \Lambda \setminus \{0\}} \frac{b_i}{b_0} \right)^{-1} = \frac{1}{b_0} \sum_{k=1}^{\infty} (-1)^k \left(\sum_{i \in \Lambda \setminus \{0\}} \frac{b_i}{b_0} \right)^k.$$

Thus $\text{Supp}(a) \subset \Lambda$. This proves that B is a local ring with maximal ideal $\mathfrak{m} \cap B$.

Now let $P(Z) \in B[Z]$, such that $P(0) \in \mathfrak{m} \cap B$ and $P'(0) \notin \mathfrak{m}$. We denote by ν the

order function of A , i.e. if $a \in A$, $a \neq 0$, $a = \sum_i a_i$ where a_i is homogeneous of degree i , $\nu(a) := \inf\{i / a_i \neq 0\}$ and the initial term of a is $\text{in}(a) := a_{\nu(a)}$. Since A is a complete local ring, it is a Henselian local ring and there exists $a \in \mathfrak{m}$ such that $P(a) = 0$. We can construct a by using the fact that

$$(4) \quad P(Z) = P(0) + P'(0)Z + Q(Z)Z^2$$

where $Q(Z) \in B[Z]$. Indeed, let Λ be a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ containing the supports of all the coefficients of $P(Z)$. In this case $a_1 := \text{in}(a) = -\frac{\text{in}(P(0))}{\text{in}(P'(0))}$ is a homogeneous element of degree $d_1 \in \Lambda$, $d_1 > 0$. If we set $P_1(Z) := P(Z + a_1)$, we see that

$$\nu(P_1(0)) = \nu(P(a_1)) > d_1.$$

Then we apply Equation (4) to $P_1(Z)$, using the fact that the coefficients of $P_1(Z)$ have support included in Λ and $P'(0) = P'_1(0)$. Thus we see that $\text{in}(a - a_1) = -\frac{\text{in}(P_1(0))}{\text{in}(P'_1(0))}$ is a homogeneous element of degree $d_2 \in \Lambda$, $d_2 > d_1$. We repeat this operation a countable number of times (since Λ is countable) in order to construct a and we see that $\text{Supp}(a) \subset \Lambda$. \square

Now we can prove the following theorem:

Theorem 4.2. *Let \mathbb{k} be a field of characteristic zero and ν an Abhyankar valuation of \mathcal{F}_n . Let $N := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $P(Z) \in V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$ (resp. $\widehat{V}_{\nu}[\langle \gamma_1, \dots, \gamma_N \rangle]$) be a monic polynomial of degree d where γ_i is a homogeneous element with respect to ν for $1 \leq i \leq N$. Then there exist integral homogeneous elements $\gamma'_1, \dots, \gamma'_N$ such that the roots of $P(Z)$ are in $V_{\nu}^{\text{fg}}[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ (resp. $\widehat{V}_{\nu}[\langle \gamma'_1, \dots, \gamma'_N \rangle]$).*

Proof. Let us prove the case $P(Z) \in V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$. Let us write $P(Z) = Z^d + a_1 Z^{d-1} + \dots + a_d$. By replacing Z by $Z - \frac{1}{d}a_1$, we may assume that $a_1 = 0$. Let i_0 be an integer such that

$$\frac{\nu(a_{i_0})}{i_0} \leq \frac{\nu(a_i)}{i}, \quad 2 \leq i \leq d.$$

Let γ be a i_0 -root of $\text{in}_{\nu}(a_{i_0})$, i.e. γ is a homogeneous element such that $\gamma^{i_0} = \text{in}_{\nu}(a_{i_0})$. By the definition of i_0 , for $2 \leq i \leq d$ we can write

$$a_i = \gamma^i a'_i$$

with $a'_i \in V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N, \gamma \rangle]$, $2 \leq i \leq d$. Then we have

$$P(\gamma Z) = \gamma^d Z^d + \gamma^{d-2} a_2 Z^{d-2} + \dots + a_d = \gamma^d (Z^d + a'_2 Z^{d-2} + \dots + a'_d)$$

Let $S(Z) := Z^d + a'_2 Z^{d-2} + \dots + a'_d$ and let $\overline{S}(Z)$ be the image of $S(Z)$ in $V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_s, \gamma \rangle] / \mathfrak{m} = \mathbb{L}$ where $\mathbb{k}_{\nu} \rightarrow \mathbb{L}$ is finite and \mathfrak{m} is the maximal ideal of $V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_s, \gamma \rangle]$. If $\overline{S}(Z) = (Z + \overline{a})^d$ where $\overline{a} \in \mathbb{L}$, since $a_1 = 0$ and $\text{char}(\mathbb{L}) = 0$, this would imply $\overline{a} = 0$. But $\overline{S}(Z) \neq Z^d$ since its coefficient of Z^{d-i_0} is non zero. Thus we can factor $\overline{S}(Z) = \overline{S}_1(Z) \overline{S}_2(Z)$ such that $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ are coprime monic polynomials in $\mathbb{L}[\gamma']$ where γ' is algebraic over \mathbb{L} , i.e. γ' is a homogeneous element of degree 0 with respect to ν . Since $V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle]$ is a Henselian local ring by Lemma 4.1, by Hensel Lemma the polynomial $S(Z)$ factors as $S(Z) = S_1(Z) S_2(Z)$ where the images of $S_1(Z)$ and $S_2(Z)$ in $V_{\nu}^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle]$ are $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ and the ν -support of the coefficients of $S_1(Z)$ and $S_2(Z)$ are contained in a finitely generated semigroup of $\mathbb{R}_{\geq 0}$.

Since $\deg_Z(S_1(Z)), \deg_Z(S_2(Z)) < d = \deg_Z(P(Z))$, the theorem is proven by induction on d by using Proposition 3.25 and Remark 3.17.

The case $P(Z) \in \widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$ is proven in a similar way by using the fact that $\widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle]$ is a complete local ring, thus a Henselian local ring. \square

Corollary 4.3. *The field $\overline{\mathbb{K}}_\nu^{\text{fg}}$ (resp. $\overline{\mathbb{K}}_\nu$) is algebraically closed and it is the algebraic closure of $\mathbb{K}_\nu^{\text{fg}}$ (resp. $\widehat{\mathbb{K}}_\nu$).*

Proof. Let $P(Z) \in \overline{\mathbb{K}}_\nu^{\text{fg}}[Z]$ be an irreducible polynomial. By multiplying $P(Z)$ by an element of V_ν , we may assume that $P(Z) \in V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$ for some homogeneous elements $\gamma_1, \dots, \gamma_N$ with respect to ν . Let us denote $P(Z) = a_d Z^d + \dots + a_0$, $a_i \in V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N \rangle]$, $0 \leq i \leq d$. Let $a \in V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N \rangle]$, $\nu(a) > 0$. Let us denote $Q(Z) := a_d^{d-1} a^d P(Z/aa_d)$. Then $Q(Z)$ is a monic polynomial of $V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$ and if z is a root of $Q(Z)$, then $\frac{z}{aa_d}$ is a root of $P(Z)$. Hence the result comes from Theorem 4.2.

The assertion concerning $\overline{\mathbb{K}}_\nu$ is proven similarly. \square

We have the similar result for $\overline{\mathbb{K}}^{\text{alg}}$:

Lemma 4.4. *The algebraic closure of \mathbb{K}_n in $\overline{\mathbb{K}}_\nu$ is equal to $\overline{\mathbb{K}}_\nu^{\text{alg}}$. In particular $\overline{\mathbb{K}}^{\text{alg}}$ is algebraically closed.*

Proof. Let $\gamma_1, \dots, \gamma_s$ be homogeneous elements with respect to ν . Let us denote by q_{i+1} the degree of the minimal polynomial of γ_{i+1} over $\mathbb{K}_n[\gamma_1, \dots, \gamma_i]$ for $0 \leq i \leq s-1$. Thus any element z of $\widehat{\mathbb{K}}_\nu[\gamma_1, \dots, \gamma_s]$ can be uniquely written as $\sum_{i \in I} A_{i_1, \dots, i_s} \gamma_1^{i_1} \dots \gamma_s^{i_s} \in \overline{\mathbb{K}}_\nu$ where $A_{i_1, \dots, i_s} \in \widehat{\mathbb{K}}_\nu$ for all $i \in I$ and $I = \{0, \dots, q_1 - 1\} \times \dots \times \{0, \dots, q_s - 1\}$.

In order to prove the lemma we need to show that, if we assume that z is algebraic over \mathcal{F}_n then $A_{i_1, \dots, i_s} \in \overline{\mathbb{K}}_\nu^{\text{alg}}$ for any i_1, \dots, i_s . In this case let $\mathbb{L} := \overline{\mathbb{K}}_\nu[\gamma_1, \dots, \gamma_{s-1}]$ and let us

write $z := \sum_{i=0}^{q_s-1} B_i \gamma_s^i$ where $B_i \in \mathbb{L}$ for all i . Let us denote $\zeta_1 := \gamma_s$ and let $\zeta_2, \dots, \zeta_{q_s}$ be the

conjugates of ζ_1 over \mathbb{K}_n . Let us denote $z_j = \sum_{i=0}^{q_s-1} B_i \zeta_j^i$ for $1 \leq j \leq q_s$. Then we have

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{q_s} \end{pmatrix} = \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{q_s-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{q_s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_{q_s} & \dots & \zeta_{q_s}^{q_s-1} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{q_s-1} \end{pmatrix}.$$

Let us denote $M := \begin{pmatrix} 1 & \zeta_1 & \dots & \zeta_1^{q_s-1} \\ 1 & \zeta_2 & \dots & \zeta_2^{q_s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \zeta_{q_s} & \dots & \zeta_{q_s}^{q_s-1} \end{pmatrix}$. This matrix is invertible and its entries are

algebraic over $\mathbb{k}(\mathbf{x})$, z_j is algebraic over \mathcal{F}_n for all j , hence B_j is algebraic over \mathcal{F}_n for all j . By induction on s we see that $A_{i_1, \dots, i_s} \in \overline{\mathbb{K}}_\nu^{\text{alg}}$ for any i_1, \dots, i_s . \square

We can summarize the situation with the following commutative diagram where the bottom part corresponds to the quotient fields of rings of the upper part and all the morphisms

are injective:

$$\begin{array}{ccccccccc}
 \mathcal{F}_n & \longrightarrow & V_\nu & \longrightarrow & V_\nu^{\text{alg}} & \longrightarrow & V_\nu^{\text{fg}} & \longrightarrow & \widehat{V}_\nu \\
 & & \downarrow & & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & \mathbb{K}_n & \longrightarrow & \mathbb{K}_\nu^{\text{alg}} & \longrightarrow & \mathbb{K}_\nu^{\text{fg}} & \longrightarrow & \widehat{\mathbb{K}}_\nu \\
 & & & & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
 & & & & \mathbb{K}_\nu^{\text{alg}} & \longrightarrow & \mathbb{K}_\nu^{\text{fg}} & \longrightarrow & \mathbb{K}_\nu
 \end{array}$$

Example 4.5. Let $g(T) = \sum_{i=1}^{\infty} c_i T^i \in \mathbb{Q}[[T]]$ be a formal power series which is not algebraic over $\mathbb{Q}[T]$. Let $\alpha := (\alpha_1, \alpha_2) \in \mathbb{N}^n$. Let us set

$$f := g\left(\frac{x_2^{\alpha_1}}{x_1^{\alpha_2}}\right) = \sum_{i=1}^{\infty} c_i \frac{x_2^{\alpha_1 i}}{x_1^{\alpha_2 i}} \in \mathbb{k}((x_1))((x_2)).$$

But $f \notin \overline{\mathbb{K}}_{\nu_\alpha}$: let $P(Z) = a_0(\mathbf{x})Z^d + \dots + a_d(\mathbf{x}) \in \widehat{V}_{\nu_\alpha}[Z]$ be a polynomial such that $P(f) = 0$. Let us write $a_i(\mathbf{x}) = \sum_{k=0}^{\infty} a_{i,k}(\mathbf{x})$ where $a_{i,k}(\mathbf{x})$ is a weighted homogeneous rational fraction of degree k if x_1 has degree α_1 and x_2 has degree α_2 . By homogeneity we have

$$a_{0,k}f^d + a_{1,k}f^{d-1} + \dots + a_{d,k} = 0 \quad \forall k \in \mathbb{N}.$$

This implies that

$$a_{0,k}(1, T)g(T^{\alpha_1})^d + a_{1,k}(1, T)g(T^{\alpha_1})^{d-1} + \dots + a_{d,k}(1, T) = 0 \quad \forall k \in \mathbb{N}.$$

Thus $a_{i,k}(\mathbf{x}) = 0$ for all $0 \leq i \leq d$ and $0 \leq k$. Hence $P(Z) = 0$ and $f \notin \overline{\mathbb{K}}_{\nu_\alpha}$.

On the other hand, $h := g\left(\frac{x_1^{2\alpha_2}}{x_2^{\alpha_1}}\right) = \sum_{i=1}^{\infty} c_i \frac{x_1^{2\alpha_2 i}}{x_2^{\alpha_1 i}} \in \widehat{\mathbb{K}}_{\nu_\alpha}$ but h is not algebraic over $\mathbb{k}((x_1))((x_2))$.

Proposition 4.6. Let $P(Z) \in V_\nu^{\text{fg}}[Z]$ (resp. $V_\nu^{\text{alg}}[Z]$) be an irreducible monic polynomial. Then $P(Z)$ is irreducible in $\widehat{V}_\nu[Z]$.

Proof. By Corollary 4.3, $P(Z)$ splits in $V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_s \rangle]$ for some homogeneous elements $\gamma_1, \dots, \gamma_s$ with respect to ν . Since $V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_s \rangle] \cap \widehat{V}_\nu = V_\nu^{\text{fg}}$ the result follows.

The proof is the same for V_ν^{alg} . \square

Lemma 4.7. Let σ be a $\widehat{\mathbb{K}}_\nu$ -automorphism of $\overline{\mathbb{K}}_\nu$. Then for any $z \in \overline{\mathbb{K}}_\nu$, we have $\nu(\sigma(z)) = \nu(z)$.

Proof. Let $z \in \widehat{\mathbb{K}}_\nu[\gamma_1, \dots, \gamma_s]$ where $\gamma_1, \dots, \gamma_s$ are homogeneous elements with respect to ν . Let us write $z := \sum_{i \in \Lambda} z_i$ where z_i is homogeneous of degree i and Λ is countable subset of \mathbb{R} with no accumulation point (see Remark 3.3). If $i_0 = \nu(z)$, then $z_{i_0} \neq 0$ and $\nu(z_i) = 0$ for all $i < i_0$. Then $\sigma(z) = \sum_i \sigma(z_i)$, for all i $\sigma(z_i)$ is homogeneous of degree i and $\sigma(z_i) = 0$ if and only if $z_i = 0$. This proves that $i_0 = \nu(\sigma(z))$. \square

Corollary 4.8. Let $P(Z) \in \widehat{V}_\nu[Z]$ be an irreducible monic polynomial. Then the Newton polygon of $P(Z)$ has only one edge. The result remains valid if we replace \widehat{V}_ν by V_ν^{alg} or V_ν^{fg} .

Proof. Let z be a root of $P(Z)$ in \overline{V}_ν . Let σ be a $\widehat{\mathbb{K}}_\nu$ -automorphism of $\overline{\mathbb{K}}_\nu$. Then $\nu(\sigma(z)) = \nu(z)$ by Lemma 4.7. The finite product of the distinct linear forms $Z - \sigma(z)$ obtained in this way is a monic polynomial with coefficients in $\widehat{\mathbb{K}}_\nu$ and divides $P(Z)$. Since $P(Z)$ is irreducible, both polynomials are equal. This proves that all the roots of $P(Z)$ have same valuation, hence the Newton polygon of $P(Z)$ has only one edge. The cases V_ν^{alg} and V_ν^{fg} are deduced from Lemma 4.6. \square

Example 4.9. Let $P(Z) := Z^3 + 3x_1x_2Z - 2x_1^4 \in \mathbb{k}[[x_1, x_2]][Z]$. We see that $P(Z)$ has a root of order 2 and two roots of order 1 in $\overline{V}_{\text{ord}}^{\text{fg}}$. By Corollary 4.8, $P(Z)$ has at least one root in $V_{\text{ord}}^{\text{alg}}$ of order 2.

Let $\sqrt{1+U} := 1 + \sum_{i \geq 1} a_i U^i$, $a_i \in \mathbb{Q}$ for all i , the formal powers series whose square is equal to $1+U$, and let $\sqrt[3]{1+U} := 1 + \sum_{i \geq 1} b_i U^i$, $b_i \in \mathbb{Q}$ for all i , the formal power series whose cube is equal to $1+U$. Then the roots of $P(Z)$ are

$$a\sqrt[3]{q + \sqrt{q^2 + p^3}} + b\sqrt[3]{q - \sqrt{q^2 + p^3}}$$

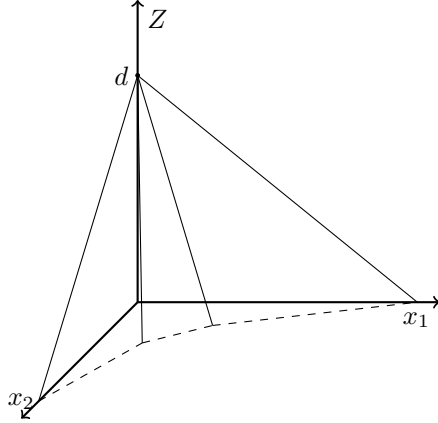
with $(a, b) = (1, 1)$, (j, j^2) or (j^2, j) and $p = x_1x_2$ and $q = x_1^4$. But

$$\sqrt[3]{q + \varepsilon\sqrt{q^2 + p^3}} = \sqrt[3]{x_1^4 + \varepsilon\sqrt{x_1^3x_2^3 + x_1^8}} = \sqrt[3]{\varepsilon}\sqrt{x_1x_2} + \eta$$

where $\varepsilon = 1$ or -1 and $\text{ord}(\eta) > 1$. Both order 1 roots of $P(Z)$ have initial term of the form $\alpha\sqrt{x_1x_2}$ where $\alpha \in \mathbb{C}^*$. Thus $P(Z)$ has only one root in $V_{\text{ord}}^{\text{alg}}$ and even in $\mathbb{K}_{\text{ord}}^{\text{fg}}$.

Let z be the only root of $P(Z)$ in $V_{\text{ord}}^{\text{alg}}$. If $z \in \mathbb{K}_n$, since $P(Z)$ is monic and \mathcal{F}_n is an integral domain, then $z \in \mathcal{F}_n$. But $\text{in}(z) = \frac{2}{3}\frac{x_1^3}{x_2} \notin \mathcal{F}_n$. Thus $z \notin \mathbb{K}_n$, hence $P(Z)$ is irreducible in $\mathbb{K}_n[Z]$. This shows that $\mathbb{K}_n \rightarrow \mathbb{K}_{\text{ord}}^{\text{alg}}$ is not a normal extension in general.

Corollary 4.10. Let $P(Z) := Z^d + a_1(\mathbf{x})Z^{d-1} + \dots + a_d(\mathbf{x}) \in \mathcal{F}_n[Z]$ be an irreducible polynomial having its roots in $\mathbb{k}[[x_1^{\frac{1}{e}}, \dots, x_n^{\frac{1}{e}}]]$ for some positive integer e . Then the Newton polyhedron of $P(Z)$ is the convex hull of the cone of \mathbb{N}^{n+1} centered in $(0, \dots, 0, d)$ and generated by the convex hull of the Newton polyhedra of $a_d(\mathbf{x})$ in \mathbb{N}^n .



Proof. Let $\alpha \in \mathbb{N}^n$. Let $z_1, \dots, z_d \in \mathbb{k}[[x_1^{\frac{1}{e}}, \dots, x_n^{\frac{1}{e}}]]$ be the roots of $P(Z)$. Then $z_i \in \widehat{V}_{\nu_\alpha}[x_1^{\frac{1}{e}}, \dots, x_n^{\frac{1}{e}}]$ for any i , the $x_i^{\frac{1}{e}}$ being homogeneous elements with respect to ν_α . Let $G \simeq (\mathbb{Z}/e\mathbb{Z})^n$ be the Galois group of the extension $\widehat{V}_{\nu_\alpha} \rightarrow \widehat{V}_{\nu_\alpha}[x_1^{\frac{1}{e}}, \dots, x_n^{\frac{1}{e}}]$. The z_i 's are

conjugated under the action of G , thus $P(Z) := \prod_{i=1}^d (Z - z_i)$ is irreducible in $\widehat{V}_{\nu_\alpha}[Z]$. This being true for any $\alpha \in \mathbb{N}^n$, the result follows from Corollary 4.8. \square

We finish this section by giving two results relating the roots of a polynomial $P(Z)$ to the roots of polynomials approximating $P(Z)$. First of all, let us consider one polynomial $P(Z)$ where A is integral domain and let ν be a valuation on A . We will write

$$\nu(P(Z)) \geq c$$

if and only if $\nu(a) \geq c$ for any coefficient a of $P(Z)$.

The following proposition is the analogue of Proposition 2.6 of [To]:

Proposition 4.11. *Let $P(Z) \in V_\nu^{fg}[Z]$ be a monic polynomial of degree d with no multiple factor. Let us write $P(Z) = P_1(Z) \dots P_r(Z)$ where $P_i(Z) \in V_\nu^{fg}[Z]$, $1 \leq i \leq r$, are irreducible monic polynomials. Let $Q(Z) \in V_\nu^{fg}[Z]$ be a monic polynomial of degree d . Let us write $P(Z) = Z^d + a_1 Z^{d-1} + \dots + a_d$, $Q(Z) = Z^d + b_1 Z^{d-1} + \dots + b_d$, and let z_1, \dots, z_d be the roots of $P(Z)$. If*

$$\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\} > d \max_{i \neq j} \{\nu(z_i - z_j)\}$$

then we have the decomposition $Q(Z) = Q_1(Z) \dots Q_r(Z)$ such that $Q_i(Z) \in V_\nu^{fg}[Z]$ is an irreducible monic polynomial, $1 \leq i \leq r$, and

$$\nu(Q_i(Z) - P_i(Z)) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d}.$$

The result is still valid if we replace V_ν^{fg} by V_ν^{alg} or \widehat{V}_ν .

Proof. Since $P(Z)$ has no multiple factor and since $\text{char}(\mathbb{k}) = 0$, we have $z_i \neq z_j$ for all $i \neq j$. Let us set $r := \max_{i \neq j} \{\nu(z_i - z_j)\}$. Let z'_i , $1 \leq i \leq d$, be the roots of $Q(Z)$. Let z be a root of $P(Z)$ in $V_\nu^{fg}[\langle \gamma_1, \dots, \gamma_N \rangle]$. Then

$$\prod_{1 \leq i \leq d} (z - z'_i) = Q(z) = Q(z) - P(z) = \sum_{i=1}^d (b_i - a_i) z^{d-i}.$$

Thus there exists at least one i such that $\nu(z'_i - z) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d} > r$. Let t be another root of $P(Z)$. Then

$$\nu(z'_i - t) = \nu(z'_i - z + z - t) = \nu(z - t) \leq r$$

since $\nu(z'_i - z) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d} > r \geq \nu(z - t)$. Thus for any root of $P(Z)$ denoted by z , there is only one i such that $\nu(z - z'_i) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d}$.

If σ is a $\mathbb{K}_\nu^{\text{fg}}$ -automorphism of $\overline{\mathbb{K}}_\nu^{\text{fg}}$, then $\sigma(z)$ is a root of $P(Z)$ and $\nu(\sigma(z'_i) - \sigma(z)) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d}$ by Lemma 4.7.

Let $\sigma_1(z), \dots, \sigma_e(z)$ be the conjugates of z over V_ν^{fg} . Let $R(Z) := (Z - z) \prod_{j=1}^e (Z - \sigma_j(z)) \in V_\nu^{\text{fg}}[Z]$. Then $R(Z)$ is an irreducible factor of $P(Z)$. Moreover $\sigma_1(z'_i), \dots, \sigma_e(z'_i)$ are conjugates of z'_i over V_ν^{fg} . Let σ be a $\mathbb{K}_\nu^{\text{fg}}$ -automorphism of $\overline{\mathbb{K}}_\nu^{\text{fg}}$. Then $\sigma(z)$ is a conjugate of z thus there exists j such that $\sigma(z) = \sigma_j(z)$. But we have

$$\nu(\sigma(z'_i) - \sigma_j(z)) = \nu(\sigma(z'_i) - \sigma(z)) = \nu(z'_i - z) = \nu(\sigma_j(z'_i) - \sigma_j(z)) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d}$$

and since there is only one root of $Q(Z)$ whose difference with $\sigma_j(z)$ has valuation greater than $\frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d}$, we necessarily have $\sigma(z'_i) = \sigma_j(z'_i)$. Thus $\sigma_1(z'_i), \dots, \sigma_e(z'_i)$ are the conjugates of z'_i over V_ν^{fg} . Thus the polynomial

$$S(Z) := (Z - z'_i) \prod_{j=1}^e (Z - \sigma_j(z'_i))$$

is irreducible in $V_\nu^{\text{fg}}[Z]$ and

$$\nu(S(Z) - R(Z)) \geq \frac{\min_{1 \leq i \leq d} \{\nu(a_i - b_i)\}}{d}.$$

The proof for \widehat{V}_ν is the same and the case V_ν^{alg} is proven with the help of Lemma 4.6. \square

Remark 4.12. Let us remark that $\nu(Q(Z) - P(Z)) > \frac{d}{2}\nu(\Delta_P)$ where Δ_P is the discriminant of $P(Z)$ implies that $\nu(Q(Z) - P(Z)) > d \max_{i \neq j} \{\nu(z_i - z_j)\}$.

Remark 4.13. This result is not true if $P(Z)$ has multiple factors. For example, let ν be a divisorial valuation and let us consider $P(Z) = Z^2$ and let $Q(Z) = X^2 + a$ where $\nu(a) = 2k + 1$ and $k \in \mathbb{N}$. Since $\nu(a)$ is odd and since the value group of ν is \mathbb{Z} , then it is not a square in \widehat{V}_ν and $Q(Z)$ is irreducible but P is not irreducible.

Proposition 4.14. *Let ν be an Abhyankar valuation and let $N := \dim_{\mathbb{Q}} \Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $P(Z) \in \widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$ be a polynomial of degree d where $\gamma_1, \dots, \gamma_N$ are homogeneous elements with respect to ν . Then there exist $\gamma'_1, \dots, \gamma'_N$ integral homogeneous elements with respect to ν and $c \in \mathbb{R}_{>0}$ such that the roots of $P(Z)$ are in $\widehat{V}_\nu[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ and for any monic polynomial $Q(Z) \in \widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N \rangle][Z]$ such that $\nu(P(Z) - Q(Z)) \geq c$, the roots of $Q(Z)$ are in $\widehat{V}_\nu[\langle \gamma'_1, \dots, \gamma'_N \rangle]$.*

Proof. The proof of this proposition is based on the proof of Theorem 4.2. So let us use the notations of that proof. Let us write $Q(Z) = Z^d + b_1 Z^{d-1} + \dots + b_d$ and let us define $R(Z) := Z^d + b'_1 Z^{d-1} + \dots + b'_d$ where $b'_i := \frac{b_i}{\gamma^i}$ for $1 \leq i \leq d$. We have $Q(\gamma Z) = \gamma^d R(Z)$. Let us assume that $\nu(b'_i - a'_i) > 0$ for all $1 \leq i \leq d$ (i.e. if $\nu(b_i - a_i) > \nu(\gamma^i)$ for all i , thus we assume here that $c > d\nu(\gamma)$). Then $\overline{R}(Z) = \overline{S}(Z)$ ($\overline{R}(Z)$ denotes the image of $R(Z)$ in $\mathbb{L}[Z]$) and the factorization $\overline{R}(Z) = \overline{S}_1(Z)\overline{S}_2(Z)$ lifts to a factorization $R(Z) = R_1(Z)R_2(Z)$ of $R(Z)$ as the product of two monic polynomials as in the proof of Theorem 4.2.

Lemma 4.15. *In the previous situation there exist two constants $a > 0$, $b \geq 0$ depending only on $S_1(Z)$ and $S_2(Z)$ such that for any $c > \max\{b, \nu(\gamma^d)\}$, we have $\nu(R_i(Z) - S_i(Z)) > \frac{c-b}{a}$ for $i = 1, 2$.*

Proof of Lemma 4.15. Let us denote by $r_{i,k}$ the coefficient of Z^k of the polynomial $R_i(Z)$, for $i = 1, 2$ and $0 \leq k \leq \deg_Z(R_i(Z))$, and let us denote by r the vector whose coordinates are the $r_{i,k}$'s. The coefficient of Z^k of $R_1(Z)R_2(Z) - S_1(Z)S_2(Z)$, for $0 \leq k \leq d$, is a polynomial $f_k(r)$ whose coefficients are in $\widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle]$ and depend themselves on the coefficients of $S(Z)$. By Theorem 1.2 [M-B], there exist $a > 0$, $b \geq 0$ such that

$$\begin{aligned} \forall c > b, \forall r \in \widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle]^{d+2} \text{ such that } \nu(f_k(r)) \geq c \quad \forall k \\ \exists r' \in \widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle]^{d+2} \text{ such that } f_k(r') = 0 \quad \forall k \\ \text{and } \nu(r'_{i,j} - r_{i,j}) \geq \frac{c-b}{a} \quad \forall i, j. \end{aligned}$$

Let us denote by $R'_i(Z)$ the polynomial whose coefficients are the $r'_{i,j}$'s where $0 \leq j \leq \deg(R_i)$. Then $R'_1(Z)R'_2(Z) = S_1(Z)S_2(Z)$. Moreover $\overline{R}'_i(Z) = \overline{R}_i(Z) = \overline{S}_i(Z)$ if $\frac{c-b}{a} > 0$. Since the roots of $\overline{S}_1(Z)$ and $\overline{S}_2(Z)$ are different, and since $\widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_N, \gamma, \gamma' \rangle][Z]$ is a GCD domain, then $R'_i(Z) = S_i(Z)$ for $i = 1, 2$. This proves the lemma. \square

Here we remark that, the constants $a, b, \nu(\gamma)$ depend only $P(Z)$. Thus the result is proven by induction on the degree of $P(Z)$ (since $\deg(S_i(Z)) < \deg(P(Z))$ for $i = 1, 2$) and using Proposition 3.25 and Remark 3.17. \square

5. MONOMIAL VALUATION CASE: GROWTH OF THE DENOMINATORS

We will first construct a subring of V_ν^{fg} containing V_ν^{alg} when ν is a monomial valuation.

Definition 5.1. Let $\alpha \in \mathbb{R}_{>0}^n$ and let δ be a (α) -homogeneous polynomial of degree d . We denote

$$\mathcal{V}_{\alpha, \delta} := \left\{ A \in \widehat{V}_{\nu_\alpha} / \exists \Lambda \text{ a finitely generated semigroup of } \mathbb{R}_{\geq 0}, \forall i \in \Lambda \exists a_i \in \mathbb{k}[\mathbf{x}] \text{ } (\alpha)\text{-homogeneous,} \right. \\ \left. \exists a \geq 0, b \in \mathbb{R} \forall i \in \Lambda \exists m(i) \in \mathbb{N} \text{ s.t. } m(i) \leq ai + b, \nu_\alpha \left(\frac{a_i}{\delta^{m(i)}} \right) = i \text{ and } A = \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}} \right\}.$$

With this notation we say that $i \mapsto ai + b$ is a *bounding function* for $\sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}}$.

By Lemma 3.10 we have $\mathbb{k}[\mathbf{x}] \subset \mathcal{V}_{\alpha, \delta} \subset V_{\nu_\alpha}^{\text{fg}}$, by identifying a formal power series $\sum_{\beta \in \mathbb{N}_{>0}^n} c_\beta x^\beta$

to $\sum_{i \in \Lambda} \frac{a_i(x)}{\delta(x)^{m(i)}}$ with $a_i(x) := \sum_{\alpha_1 \beta_1 + \dots + \alpha_n \beta_n = i} c_\beta x^\beta$ et $m(i) = 0$ for all $i \in \Lambda$. We extend

in an obvious way the addition and multiplication of $\mathbb{k}[\mathbf{x}]$ to $\mathcal{V}_{\alpha, \delta}$: this defines a \mathbb{k} -algebra structure over $\mathcal{V}_{\alpha, \delta}$. We can see easily that if $i \mapsto ai + b$ is a bounding function of A and $B \in \mathcal{V}_{\alpha, \delta}$ then it is also a bounding function of $A + B$ and AB .

Definition 5.2. Let $A := \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}} \in \mathcal{V}_{\alpha, \delta}$, $A \neq 0$. Let i_0 be the least element of Λ such that $a_{i_0} \neq 0$. We say that $\frac{a_{i_0}}{\delta^{m(i_0)}}$ is the initial term of A with respect to ν_α or its (α) -initial term. We denote it by $\text{in}_\alpha(A)$.

Lemma 5.3. Let δ and δ' be two (α) -homogeneous polynomials. We have the following properties:

- i) The (α) -homogeneous irreducible divisors of δ divide δ' if and only if $\mathcal{V}_{\alpha, \delta'} \subset \mathcal{V}_{\alpha, \delta}$. We denote by \mathcal{V}_α the inductive limit of the $\mathcal{V}_{\alpha, \delta}$.
- ii) The valuation ν_α is well defined over $\mathcal{V}_{\alpha, \delta}$ and extends to \mathcal{V}_α . Its valuation ring is exactly \mathcal{V}_α : \mathcal{V}_α is a finitely generated valuation ring.

Proof. It is clear that if the irreducible divisors of δ divide δ' then $\mathcal{V}_{\alpha, \delta} \subset \mathcal{V}_{\alpha, \delta'}$. On the other hand if $\mathcal{V}_{\alpha, \delta} \subset \mathcal{V}_{\alpha, \delta'}$, then $\frac{1}{\delta} \in \mathcal{V}_{\alpha, \delta'}$, thus there exist a (α) -homogeneous polynomial $a \in \mathbb{k}[\mathbf{x}]$ and an integer $m \in \mathbb{N}$ such that $\frac{1}{\delta} = \frac{a}{\delta'^m}$, hence $a\delta = \delta'^m$. This proves i).

If $A \in \mathcal{V}_{\alpha, \delta}$ and $B \in \mathcal{V}_{\alpha, \delta'}$ satisfy $\nu_\alpha(B) \geq \nu_\alpha(A)$, by denoting by $\frac{a_0(x)}{\delta(x)^{m(0)}}$ the first non-zero term in the expansion of A , we can check easily that $\frac{B}{A} \in \mathcal{V}_{\alpha, \delta\delta'^{a_0}}$. This proves ii). \square

Definition 5.4. For any $\alpha \in \mathbb{R}_{>0}^n$ we denote by \mathcal{K}_α the fraction field of \mathcal{V}_α and

$$\overline{\mathcal{K}}_\alpha := \lim_{\substack{\longrightarrow \\ \gamma_1, \dots, \gamma_s}} \mathcal{K}_\alpha[\langle \gamma_1, \dots, \gamma_s \rangle]$$

the limit being taken over all subsets $\{\gamma_1, \dots, \gamma_s\}$ of (integral) homogeneous elements with respect to ν .

If $\gamma_1, \dots, \gamma_s$ are homogeneous elements with respect to ν_α we denote by $\mathcal{V}_{\alpha, \delta}[\langle \gamma_1, \dots, \gamma_s \rangle]$ the ring of elements $\sum_{\underline{k}} A_{\underline{k}} \gamma^{\underline{k}}$ where the sum is finite, $\underline{k} := (k_1, \dots, k_s)$, $A_{\underline{k}} = \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}}$ where $a_i \in \mathbb{k}[x]$ is (α) -homogeneous, there exist two constants $a \geq 0$, $b \in \mathbb{R}$ such that $m(i) \leq ai + b$ for all i and there exists $i_0 \in \Lambda$ such that $\nu_\alpha(\frac{a_i}{\delta^{m(i)}}) = i - i_0$ and $\nu(\gamma^{\underline{k}}) \geq i_0$.

Then we show the following version of the Implicit Function Theorem inspired by Lemma 1.2 [Ga] (see also Lemma 2.2. [To]):

Proposition 5.5. *Let $\alpha \in \mathbb{R}_{>0}^n$ and let $P(Z) \in \mathcal{V}_{\alpha, \delta}[\langle \gamma_1, \dots, \gamma_s \rangle][Z]$, $P(Z) = \sum_{k=0}^d a_k Z^k$ where γ_i is homogeneous for all i with respect to ν_α . Let $u \in \mathcal{V}_{\alpha, \delta}[\langle \gamma_1, \dots, \gamma_s \rangle]$ such that $\nu_\alpha(P(u)) > 2\nu_\alpha(P'(u))$. Let $\frac{\tilde{\delta}}{\delta^m}$ denote the initial term of $P'(u)$ with respect to ν_α . Then there exists a unique solution \bar{u} in $\mathcal{V}_{\alpha, \delta \tilde{\delta}}[\langle \gamma_1, \dots, \gamma_s \rangle]$ of $P(Z) = 0$ such that*

$$\nu_\alpha(\bar{u} - u) \geq \nu_\alpha(P(u)) - \nu_\alpha(P'(u)).$$

Proof. Set $Z = u + T$. We are looking for a solution t of equation $P(u + T) = 0$ such that $\nu_\alpha(t) \geq \nu_\alpha(P(u)) - \nu_\alpha(P'(u)) (> \nu_\alpha(P'(u)))$. We have

$$P(u + T) = P(u) + P'(u)T + Q(T)T^2 = 0$$

where $Q(T) \in \mathcal{V}_{\alpha, \delta}[\langle \gamma_1, \dots, \gamma_s \rangle][T]$. The valuation ν_α defined on the ring $\mathcal{V}_{\alpha, \delta}[\langle \gamma_1, \dots, \gamma_s \rangle]$ takes values in a sub-group Γ of \mathbb{R} . Let us denote by V the valuation ring associated to ν_α and let us denote by \widehat{V} its completion. Let V^{fg} be the subring of \widehat{V} of all elements of \widehat{V} whose ν_α -support is included in a finitely generated semigroup. Then V^{fg} is a Henselian local ring by Lemma 4.1. Let us denote $T = \frac{\tilde{\delta}}{\delta^m} Y$. Thus we are looking for solving the following equation:

$$P(u) + P'(u) \frac{\tilde{\delta}}{\delta^m} Y + Q\left(\frac{\tilde{\delta}}{\delta^m} Y\right) \frac{\tilde{\delta}^2}{\delta^{2m}} Y^2 = 0$$

or

$$S(Y) := A + BY + R(Y)Y^2 = 0$$

where $A := \frac{\delta^{2m}}{\tilde{\delta}^2} P(u) \in V^{\text{fg}}$, $B \in V^{\text{fg}}$ is invertible and its initial term is equal to 1, and $R(Y) := Q\left(\frac{\tilde{\delta}}{\delta^m} Y\right) \in V^{\text{fg}}[Y]$. By Hensel Lemma this equation has a unique solution $y \in V^{\text{fg}}$ such that $\nu_\alpha(y) > 0$. Let Λ be a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$ containing the ν_α -support of y , A , B and the coefficients of $R(Y)$. Since Λ is countable we may write $\Lambda = \{i_1, i_2, i_3, \dots\}$ with $i_k < i_{k+1}$ for any $k \in \mathbb{N}$. We may write $y = \sum_{k=1}^\infty y_{i_k}$ is homogeneous of degree $i_k \in \Lambda$ with respect to ν_α such that $i_{k+1} > i_k$ for all $k \geq 1$. For any $f \in V^{\text{fg}}$ let us denote by $\text{in}_\alpha(f)$ its initial term. We may construct this solution y by induction:

necessarily we have $y_{i_1} = -\text{in}_\alpha(A)$ since $\text{in}(B) = 1$, and $y_{i_{l+1}} = -\text{in}_\alpha\left(S\left(\sum_{k=1}^l y_{i_k}\right)\right)$ for all $l \geq 1$. Thus we see that $y_{i_k} = \frac{c_k}{(\delta \tilde{\delta})^{m(i_k)}}$ where $c_k \in \mathbb{k}[x][\gamma_1, \dots, \gamma_s]$ and $m(i_k) \in \mathbb{N}$ for all k .

Let us write $a_k := \sum_{l \in \Lambda} a_{k,l}$ where $a_{k,l}$ is homogeneous of degree l with respect to ν_α .

For any $i \in \Lambda$ let v_i be a new variable and set $v := \sum_{i \in \Lambda} v_i$. We extend the valuation ν_α to $V^{\text{fg}}[v_i, \dots, v_i, \dots]$ by setting $\nu_\alpha(v_i) := i$ for any $i \in \Lambda$. We may write formally $P(v) = \sum_i P_i(v)$ where $P_i(v) \in \mathbb{Z}[a_{k,l}, v_j]$ is the homogeneous term of degree i with respect to ν_α . Let $\lambda := 2\nu_\alpha(P'(u))$ and let $i \in \Lambda$; we have

$$-P_{\lambda+i}(v) = \frac{\tilde{\delta}}{\delta^m} v_i + Q_i(v)$$

where $Q_i(v)$ is a polynomial with integer coefficients and variables $a_{k,l}$ ($l \leq \lambda + i$) and v_j ($j < i$). Evaluating in y , we have $P_{\lambda+i}(y) = 0$, hence $y_i = -\frac{\delta^m Q_i(\overline{u})}{\tilde{\delta}}$.

Let Q be a monomial of $Q_i(v)$ and let r be a real number such that $r > \frac{\lambda}{\deg(P)-1}$. We may write $Q = R \prod_{j=1}^{j_0} v_{k_j}$ where R does not depend on v_k for any $k \geq r$ and $k_{j_0} < i$. Let $i \rightarrow ai + b$ be a common bounding function for the a_k and $\sum_{j < r} y_j$ seen as elements of $\mathcal{V}_{\alpha, \delta\tilde{\delta}}[\langle \gamma_1, \dots, \gamma_s \rangle]$. We may do the change of variables $Z' := Z - \text{in}_\alpha(u)$ and multiply $P(Z')$ by a large power of $\delta\tilde{\delta}$ in order to assume that $m(0) \leq 0$, thus by choosing a large enough we may assume $b = 0$. The denominator of R may be written as $(\delta\tilde{\delta})^{a(i+\lambda-\sum_{j=1}^{j_0} k_j)}$ (since Q is quasi-homogeneous of total weight $i + \lambda$ if v_j has weight j and $a_{k,l}$ has weight l). Let $\beta \leq 0$ and let $\alpha \geq a$ such that

$$m(r) - \beta \leq r\alpha \quad \text{and} \quad -\beta \geq \frac{\lambda\alpha}{\deg(P) - 1}.$$

Such a couple (α, β) exists since $r > \frac{\lambda}{\deg(P)-1}$: indeed let $\varepsilon := r - \frac{\lambda}{\deg(P)-1}$. Then let us fix $\alpha \geq a$ such that $\varepsilon\alpha \geq m(r)$ and let us denote $-\beta := (r - \varepsilon)\alpha$.

We have $m(i) \leq ai \leq \alpha i + \beta$ for all $i < r$. If $i \geq r$, Let us show by induction that $m(i) \leq \alpha i + \beta$: we have

$$\begin{aligned} m(i) &\leq a \left(i + \lambda - \sum_{j=1}^{j_0} k_j \right) + \sum_{j=1}^{j_0} m(k_j) + 1 \leq a \left(i + \lambda - \sum_{j=1}^{j_0} k_j \right) + \alpha \sum_{j=1}^{j_0} k_j + j_0\beta + 1 \\ &\leq a \left(i + \lambda - \sum_{j=1}^{j_0} k_j \right) + \alpha \sum_{j=1}^{j_0} k_j + \beta \deg(P) + 1 \leq \alpha(i + \lambda) + \beta \deg(P) + 1 \leq \alpha i + \beta, \end{aligned}$$

the first inequality coming from $y_i = \frac{\delta^m Q_i(y)}{\tilde{\delta}}$, the second one coming from the induction

hypothesis, the third one coming from $\deg(P) \geq j_0$, the fourth one coming from $i + \lambda \geq \sum_{j=1}^{j_0} k_j$

and $\alpha \geq a$, and the last one from $-\beta \geq \frac{\lambda\alpha}{\deg(P) - 1}$.

Finally we see that $m(i) \leq \alpha i + \beta$. This proves the proposition. \square

We can deduce the following theorem:

Theorem 5.6. *Let \mathbb{k} be a field of characteristic zero. Let $\alpha \in \mathbb{R}_{\geq 0}^n$ and let us set $N = \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n)$. Let $P(Z) \in \mathcal{V}_\alpha[\langle \gamma_1, \dots, \gamma_s \rangle][Z]$ be a monic polynomial where $\gamma_1, \dots, \gamma_s$ are homogeneous elements with respect to ν_α . Then there exist integral homogeneous elements with respect to ν_α , denoted by $\gamma'_1, \dots, \gamma'_N$, such that $P(Z)$ has all its roots in $\mathcal{V}_\alpha[\langle \gamma'_1, \dots, \gamma'_N \rangle]$.*

Proof. First we may assume that $P(Z)$ is irreducible. Let $z \in V_{\nu_\alpha}^{\text{fg}}[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ be a root of $P(Z)$ where γ'_i is an integral homogeneous with respect to ν_α (by Theorem 4.2). Since $P(Z)$ is irreducible, then $P'(z) \neq 0$. Let $\tilde{z} \in \mathcal{V}_\alpha[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ be a truncation of z at a high order in such way that

$$\nu_\alpha(P(\tilde{z})) > 2\nu_\alpha(P'(\tilde{z})) = 2\nu_\alpha(P'(z))$$

and such that $\nu_\alpha(P(\tilde{z})) - \nu_\alpha(P'(\tilde{z}))$ is strictly greater than $\nu_\alpha(z - z')$ for any other root z' of $P(Z)$. Then we apply Proposition 5.5, and we get a root $\bar{z} \in \mathcal{V}_\alpha[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ of $P(Z)$ such that $\nu_\alpha(z - \bar{z})$ is strictly greater than $\nu_\alpha(z - z')$ for any root z' of $P(Z)$ different from z . Thus $z = \bar{z} \in \mathcal{V}_\alpha[\langle \gamma'_1, \dots, \gamma'_N \rangle]$. \square

Corollary 5.7. *The field $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ is a subfield of \mathcal{K}_α .*

Proof. Let $z \in \mathbb{K}_{\nu_\alpha}^{\text{alg}}$ and let $P(Z) = a_0 Z^d + \dots + a_d \in \mathcal{F}_n[Z]$ be a polynomial such that $P(z) = 0$. Then $a_0 z \in \mathbb{K}_{\nu_\alpha}^{\text{alg}}$ is a root of the polynomial $a_0^{d-1} P(Z/a_0) = Z^d + a_1 Z^{d-1} + a_2 a_0 Z^{d-2} + \dots + a_d a_0^{d-1}$ which is a distinguished polynomial. Hence $a_0 z \in \mathcal{V}_\alpha$ by Theorem 5.6 and $z \in \mathcal{K}_\alpha$. \square

Example 5.8. Let us assume that $\text{Disc}_Z(P(Z))$ is normal crossing after a formal change of coordinates and let us assume that \mathbb{k} is algebraically closed. This means that there exist power series $x_i(\mathbf{y}) \in (\mathbf{y})\mathbb{k}[[\mathbf{y}]]$ ($\mathbf{y} = (y_1, \dots, y_n)$), for $1 \leq i \leq n$, such that the morphism of \mathbb{k} -algebras $\varphi : \mathbb{k}[[\mathbf{x}]] \longrightarrow \mathbb{k}[[\mathbf{y}]]$ defined by $\varphi(f(\mathbf{x})) = f(x_1(\mathbf{y}), \dots, x_n(\mathbf{y}))$ is an isomorphism, and such that

$$\varphi(\text{Disc}_Z(P(Z)))\mathbb{k}[[\mathbf{y}]] = y_1^{e_1} \dots y_m^{e_m} \mathbb{k}[[\mathbf{y}]], \quad m \leq n.$$

From the Abhyankar-Jung Theorem [Ab] (or [KV], [PR]), the roots of $P(Z)$ can be written as $t_k = \sum_{\beta \in \mathbb{N}^n} t_{k,\beta} \mathbf{w}^\beta$, for $1 \leq k \leq d$, where $\mathbf{w} = (y_1^{1/e}, \dots, y_m^{1/e}, y_{m+1}, \dots, y_n)$ for some integer $e \in \mathbb{N}$. Let us denote by $f_i(\mathbf{x})$, $1 \leq i \leq n$, the power series satisfying $\varphi(f_i(\mathbf{x})) = y_i$. Let $\alpha \in \mathbb{N}^n$ and let us denote $f_i(\mathbf{x}) = l_{i,\alpha}(\mathbf{x}) + \varepsilon_{i,\alpha}(\mathbf{x})$ where $l_{i,\alpha}(\mathbf{x})$ is (α) -homogeneous and $\nu_\alpha(\varepsilon_i(\mathbf{x})) \geq \nu_\alpha(l_{i,\alpha}(\mathbf{x}))$ for any i . Thus we have for $1 \leq i \leq m$:

$$y_i^{\frac{1}{e}} = l_{i,\alpha}(\mathbf{x})^{\frac{1}{e}} \left(1 + \frac{\varepsilon_{i,\alpha}(\mathbf{x})}{l_{i,\alpha}(\mathbf{x})} \right)^{\frac{1}{e}} = l_{i,\alpha}(\mathbf{x})^{\frac{1}{e}} \left(1 + \sum_{k \leq 1} c_k \frac{\varepsilon_{i,\alpha}(\mathbf{x})^k}{l_{i,\alpha}(\mathbf{x})^k} \right)$$

where $c_k \in \mathbb{Q}$ for all k . Hence

$$y_1^{\frac{1}{e}} \dots y_m^{\frac{1}{e}} = l_{1,\alpha}(\mathbf{x})^{\frac{1}{e}} \dots l_{m,\alpha}(\mathbf{x})^{\frac{1}{e}} \prod_{j=1}^m \left(1 + \sum_{k \leq 1} c_k \frac{\varepsilon_{j,\alpha}(\mathbf{x})^k \prod_{p \neq j} l_{p,\alpha}(\mathbf{x})^k}{(\prod_{p=1}^m l_{p,\alpha}(\mathbf{x}))^k} \right).$$

We remark that $\text{Disc}_Z(P(Z)) = \prod_{p=1}^m l_{p,\alpha}(\mathbf{x})^{e_p} + \varepsilon(\mathbf{x})$ with $\nu_\alpha(\varepsilon(\mathbf{x})) > \nu_\alpha(\prod_{p=1}^m l_{p,\alpha}(\mathbf{x})^{e_p})$. Let $\gamma := \prod_{p=1}^m l_{p,\alpha}(\mathbf{x})^{\frac{1}{e}}$ be a root of $Z^e - \prod_{p=1}^m l_{p,\alpha}(\mathbf{x})$ (in particular it is an integral homogeneous element with respect to ν_α), and set $\delta := \prod_{p=1}^m l_{p,\alpha}(\mathbf{x})^{e_p}$. Here δ divides the initial term of the discriminant of $P(Z)$. We have the following three cases:

- i) If φ is a linear change of coordinates (i.e. $\alpha = (1, \dots, 1)$ and $\varepsilon_{i,\alpha} = 0 \ \forall i$), then the roots of $P(Z)$ are in $\mathbb{k}[[\mathbf{x}]][\gamma]$.
- ii) If φ is a quasi-linear change of variables (i.e. $\alpha \in \mathbb{N}^n$ and $\varepsilon_{i,\alpha} = 0 \ \forall i$), then the roots of $P(Z)$ are in $\mathbb{k}[[\mathbf{x}]][\gamma]$.
- iii) If (at least) one of the $\varepsilon_{i,\alpha}$ is not zero, then the roots of $P(Z)$ are in $\mathcal{V}_{\alpha,\delta}[\langle \gamma \rangle]$.

Example 5.9. Let $P(Z) = Z^2 + 2aZ + b$. Let δ denote the (α) -initial term of the discriminant of $P(Z)$, i.e. the (α) -initial term of $a^2 - b$. Then the roots of $P(Z)$ are of the form $-a + \sqrt{a^2 - b} \in \mathcal{V}_{\alpha, \delta}[\langle \gamma \rangle]$ where γ is a root square of δ . Here δ divides the initial term of the discriminant of $P(Z)$.

Example 5.10. Let $P(Z) = Z^3 + 3x_2^2Z - 2(x_1^3 + \varepsilon)$ where ε is a homogeneous polynomial of degree greater or equal to 4. Its discriminant is $D := x_1^6 + x_2^6 + 2x_1^3\varepsilon + \varepsilon^2$ whose initial term is $x_1^6 + x_2^6$. The roots of P are

$$a\sqrt[3]{x_1^3 + \varepsilon + \sqrt{D}} + b\sqrt[3]{x_1^3 + \varepsilon - \sqrt{D}}$$

with $(a, b) = (1, 1)$, (j, j^2) or (j^2, j) . But we have

$$\sqrt[3]{x_1^3 + \varepsilon + \sqrt{D}} = \gamma_1 \sqrt[3]{1 + \varepsilon + \frac{\gamma_2}{x_1^3 + \gamma_2} \sqrt{1 + \frac{2x_1^3\varepsilon + \varepsilon^2}{\delta} - \frac{\gamma_2}{x_1^3 + \gamma_2}}} \in \mathcal{V}_{(1,1), \delta} \left[\left\langle \gamma_1, \gamma_2, \frac{\gamma_2\varepsilon}{x_1^3 + \gamma_2} \right\rangle \right]$$

with $\gamma_2^2 = x_1^6 + x_2^6$, $\gamma_1^3 = x_1^3 + \gamma_2$ and $\delta = x_1^6 + x_2^6$ is the initial term of D . By doing the same remark for $\sqrt[3]{x_1^3 + \varepsilon - \sqrt{D}}$, we see that there exist $\gamma_1, \dots, \gamma_5$ homogeneous elements with respect to ord such that the roots of $P(Z)$ are in $\mathcal{V}_{(1,1), \delta}[\langle \gamma_1, \dots, \gamma_5 \rangle]$. Here δ divides the initial term of the discriminant of $P(Z)$, but there is no reason that the roots of $P(Z)$ are in $\mathcal{V}_{\alpha, \delta}[\langle \gamma \rangle]$ where γ is an (integral) homogeneous element with respect to ν_α .

6. APPROXIMATION OF MONOMIAL VALUATIONS BY DIVISORIAL MONOMIAL VALUATIONS

In several cases, it will be easier to work with a monomial valuation ν_α which is divisorial, i.e. such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = 1$. In order to extend some results which are proven for divisorial monomial valuations to general monomial valuations, we will approximate monomial valuations by divisorial monomial valuations. The aim of this section is to explain how this can be done.

Definition 6.1. Let $\alpha \in \mathbb{R}_{>0}^n$. Let $l_\alpha : \mathbb{Q}^n \rightarrow \mathbb{R}$ be the \mathbb{Q} -linear morphism defined by $l_\alpha(q_1, \dots, q_n) := \sum_i \alpha_i q_i$. We denote by Rel_α the kernel of this morphism. For any $\varepsilon > 0$ and $q \in \mathbb{N}$, we define the following set:

$$\text{Rel}(\alpha, q, \varepsilon) := \left\{ \alpha' \in \mathbb{N}_{>0}^n / \text{Rel}_\alpha \subset \text{Rel}_{\alpha'} \text{ and } \max_i \left| q - \frac{\alpha'_i}{\alpha_i} \right| < q\varepsilon \right\}.$$

Example 6.2. If $n = 4$, and $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$, $\alpha_3 = 13\sqrt{2} + \sqrt{3}$, $\alpha_4 = \sqrt{2} + 757\sqrt{3}$, then any α' of the form $(n_1, n_2, 13n_1 + n_2, n_1 + 757n_2)$, where $n_1, n_2 \in \mathbb{N}_{>0}$, will satisfy $\text{Rel}_\alpha \subset \text{Rel}_{\alpha'}$.

Remark 6.3. Since \mathbb{Q} is dense in \mathbb{R} , for any $\alpha \in \mathbb{R}_{>0}^n$ and any $\varepsilon > 0$ there always exists $q \in \mathbb{N}$ such that $\text{Rel}(\alpha, q, \varepsilon) \neq \emptyset$.

Moreover if $\alpha \in \mathbb{N}^n$ then $\text{Rel}(\alpha, q, \varepsilon) = \{q\alpha\}$ if $0 < \varepsilon < \frac{1}{q \max\{\alpha_i\}}$. Indeed in this case the only $\alpha' \in \mathbb{N}^n$ satisfying $\max_i |q\alpha_i - \alpha'_i| < q\alpha_i\varepsilon$ is $\alpha' = \alpha$.

Lemma 6.4. Let $\alpha, \alpha' \in \mathbb{R}_{>0}^n$. Then $\text{Rel}_\alpha \subset \text{Rel}_{\alpha'}$ if and only if every (α) -homogeneous polynomial is a (α') -homogeneous polynomial.

Moreover if $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ and if $a(\mathbf{x})$ is a (α) -homogeneous polynomial then

$$q(1 - \varepsilon)\nu_\alpha(a(\mathbf{x})) \leq \nu_{\alpha'}(a(\mathbf{x})) \leq q(1 + \varepsilon)\nu_\alpha(a(\mathbf{x})).$$

Proof. Let us assume that $\text{Rel}_\alpha \subset \text{Rel}_{\alpha'}$ and let $a(\mathbf{x})$ be a (α) -homogeneous polynomial. This means that for any $p, q \in \mathbb{N}_{>0}^n$, if \mathbf{x}^p and \mathbf{x}^q are two non zero monomials of $a(\mathbf{x})$, then $\sum_i \alpha_i p_i = \sum_i \alpha_i q_i$. In particular $p - q \in \text{Ker}(l_\alpha)$, thus $\sum_i \alpha'_i p_i = \sum_i \alpha'_i q_i$. Thus $a(\mathbf{x})$ is a (α') -homogeneous.

On the other hand let us assume that every (α) -homogeneous polynomial is a (α') -homogeneous polynomial. Let $r \in \text{Rel}_\alpha$. We can write $r = p - q$ where $p, q \in \mathbb{Q}_{>0}^n$. By multiplying r by a positive integer m , we may assume that $mp, mq \in \mathbb{N}_{>0}^n$. By assumption on r , the polynomial $\mathbf{x}^{mp} + \mathbf{x}^{mq}$ is (α) -homogeneous. Thus it is (α') -homogeneous. This means that $\sum_i \alpha'_i mp_i = \sum_i \alpha'_i mq_i$. Hence $\sum_i \alpha'_i (p_i - q_i) = 0$ and $r = p - q \in \text{Rel}_{\alpha'}$.

Now let \mathbf{x}^p be a monomial. Then

$$\nu_{\alpha'}(\mathbf{x}^p) = \sum_i \alpha'_i p_i.$$

But $q(1 - \varepsilon)\alpha_i \leq \alpha'_i \leq q(1 + \varepsilon)\alpha_i$ for any $1 \leq i \leq n$. This proves both inequalities. \square

Example 6.5. Let $\alpha \in \mathbb{N}^n$ and $\alpha' \in \mathbb{R}_{>0}^n$. Then $\text{Rel}_\alpha \subset \text{Rel}_{\alpha'}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $\alpha' = \lambda\alpha$.

Lemma 6.6. Let $\alpha \in \mathbb{R}_{>0}^n$, $A \in \mathcal{V}_{\alpha, \delta}$, $A = \sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$. For any $\varepsilon > 0$ small enough there exists $s(\varepsilon) \in \mathbb{N}$ such that for any $q \in \mathbb{N}$, any $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ and any $s \geq s(\varepsilon)$:

$$\delta(\mathbf{x})^s \sum_{i \in \Lambda} \frac{a_i}{\delta_i^{m(i)}} (x_1 \delta(\mathbf{x})^{\alpha'_1 s}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n s}) \in \mathbb{K}[\mathbf{x}].$$

Proof. Let $a(\mathbf{x}), \delta(\mathbf{x}) \in \mathbb{K}[\mathbf{x}]$ be (α) -homogeneous polynomials and let $m \in \mathbb{N}$ such that $\nu_\alpha \left(\frac{a(\mathbf{x})}{\delta(\mathbf{x})^m} \right) = i$. Let $s \in \mathbb{N}$. Then we have

$$(5) \quad \frac{a(x_1 \delta(\mathbf{x})^{\alpha'_1 s}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n s})}{\delta(x_1 \delta(\mathbf{x})^{\alpha'_1 s}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n s})^m} = a(\mathbf{x}) \delta(\mathbf{x})^{s[\nu_{\alpha'}(a(\mathbf{x})) - \nu_{\alpha'}(\delta(\mathbf{x}))m] - m}.$$

Now let $\sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}} \in \mathcal{V}_{\alpha, \delta}$ with $m(i) \leq ai + b$ for any $i \in \Lambda$, Λ being a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Set $d_\alpha := \nu_\alpha(\delta)$. Thus $\nu_\alpha(a_i) = d_\alpha m(i) + i$ for any $i \in \Lambda$. Hence

$$\begin{aligned} \nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) &\geq q(1 - \varepsilon)[d_\alpha m(i) + i] - q(1 + \varepsilon)m(i)d_\alpha \\ \nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) &\geq q(1 - \varepsilon)i - 2q\varepsilon d_\alpha m(i). \end{aligned}$$

Since $(1 - \varepsilon)i - 2\varepsilon d_\alpha m(i) \geq (1 - \varepsilon)i - 2\varepsilon d_\alpha(ai + b)$, there exists ε small enough and $a_\varepsilon > 0$ such that

$$\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) \geq a_\varepsilon i$$

for all $q \in \mathbb{N}$, all $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ and all $i \in \Lambda$, $i > 0$. Thus for s large enough

$$s\nu_{\alpha'}(a_i) - m(i)\nu_{\alpha'}(\delta) \geq ai.$$

Moreover if $s \geq b$, we obtain the result, by Equation (5). \square

Lemma 6.7. Let $\alpha \in \mathbb{R}_{>0}^n$ and let $A \in \mathcal{V}_\alpha$. Let us write

$$A = \sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$$

where Λ is a finitely generated sub-semigroup of $\mathbb{R}_{>0}$ and $m(i)$ is bounded by an affine function. Then there exists $\varepsilon_A > 0$ such that for all $0 < \varepsilon \leq \varepsilon_A$, for all $q \in \mathbb{N}$, for all

$\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$, the element $\sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$ is in the fraction field of $\mathcal{V}_{\alpha'}$.

Moreover if $A \in \mathcal{V}_{\alpha}$ is not invertible, i.e. $\nu_{\alpha}(A) > 0$, then we may even choose $\varepsilon_A > 0$ such that for all $0 \leq \varepsilon \leq \varepsilon_A$, for all $q \in \mathbb{N}$, for all $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$, $\sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \in \mathcal{V}_{\alpha'}$ and this element is not invertible in $\mathcal{V}_{\alpha'}$.

Proof. Let $a, b \geq 0$ such that $m(i) \leq ai + b$ for any $i \in \Lambda$. By Lemma 6.4, we have

$$\nu_{\alpha'} \left(\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \right) \geq q(1 - \varepsilon)i - 2q\varepsilon m(i)\nu_{\alpha}(\delta(\mathbf{x})).$$

Let ε_A be a positive real number such that $\varepsilon_A < \frac{1}{1+2a\nu_{\alpha}(\delta(\mathbf{x}))}$ and set $\eta := 1 - \varepsilon_A(1 + 2a\nu_{\alpha}(\delta(\mathbf{x})))$. Then for any $0 \leq \varepsilon \leq \varepsilon_A$, any $q \in \mathbb{N}$ and any $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ we have

$$\nu_{\alpha'} \left(\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \right) \geq \eta qi - 2qb\varepsilon\nu_{\alpha}(\delta(\mathbf{x})) \quad \forall i \in \Lambda.$$

This proves that $\sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$ is in the fraction field of $\mathcal{V}_{\alpha'}$.

If $\nu_{\alpha}(A) > 0$, then $a_0(\mathbf{x}) = 0$. Let $i_0 := \nu_{\alpha}(A)$. Let $\varepsilon \leq 0$ be such that

$$i_0 > \varepsilon((1 + 2a\nu_{\alpha}(\delta(\mathbf{x})))i_0 + 2b\nu_{\alpha}(\delta(\mathbf{x}))).$$

In this case $\nu_{\alpha'} \left(\frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \right) > 0$ for any $i \in \Lambda$, $i \geq i_0$. This proves the second assertion. \square

Corollary 6.8. *Let $A \in \mathcal{V}_{\alpha}$ be a non-invertible element, $A = \sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}}$. There exists*

$\varepsilon_A > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_A$, for any $q \in \mathbb{N}$ and any $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$, $\sum_{i \in \Lambda} \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \in \mathcal{V}_{\alpha'}$ and if it is irreducible in $\mathcal{V}_{\alpha'}$ then A is irreducible in \mathcal{V}_{α} .

Proof. The first assertion is Lemma 6.7. If A is not irreducible in \mathcal{V}_{α} , then $A = A_1 A_2$ in \mathcal{V}_{α} and A_1 and A_2 are not invertible. By Lemma 6.7, there exists $\varepsilon_A > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_A$, for any $q \in \mathbb{N}$ and any $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$, A , A_1 and A_2 are non invertible elements of $\mathcal{V}_{\alpha'}$. Thus A is not irreducible in $\mathcal{V}_{\alpha'}$. \square

When the components of α are \mathbb{Q} -linearly independent Theorem 5.6 gives the main result of [McD] using Lemma 6.6:

Theorem 6.9. [McD] *Let \mathbb{k} be a field of characteristic zero and $\alpha \in \mathbb{R}_{>0}^n$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$. Then*

$$\mathbb{K}_{\nu_{\alpha}}^{\text{alg}} \subset \bigcup_{\sigma} \mathbb{k}((x^{\beta}, \beta \in \sigma \cap \mathbb{Z}^n))$$

where the first union runs over all rational strictly convex cones σ such that $\langle \alpha, \tau \rangle > 0$ for any $\tau \in \sigma$, $\tau \neq 0$. Moreover we have:

$$\overline{\mathbb{K}}_{\nu_{\alpha}} \subset \bigcup_{\sigma} \bigcup_{\mathbb{k}'} \bigcup_{q \in \mathbb{N}} \mathbb{k}' \left(\left(x^{\beta}, \beta \in \sigma \cap \frac{1}{q} \mathbb{Z}^n \right) \right)$$

where the first union runs over all rational strictly convex cones σ such that $\langle \alpha, \tau \rangle > 0$ for any $\tau \in \sigma$, $\tau \neq 0$, and the second union runs over all the fields \mathbb{k}' finite over \mathbb{k} .

Proof. The only (α) -homogeneous polynomials are the monomials. Let $\beta \in \mathbb{N}^n$ and A be an element of $\mathcal{V}_{\alpha, \mathbf{x}^\beta} : A = \sum_{i \in \Lambda} \frac{\mathbf{x}^{p(i)}}{\mathbf{x}^{m(i)\beta}}$ where Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. By Lemma 6.6, we see that the monomial map defined by $x_j \mapsto x_j \mathbf{x}^{s\alpha'_j \beta}$ maps A onto an element of $\mathbb{k}[[\mathbf{x}]]$ for $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$, $\varepsilon > 0$ small enough and s large enough. Such a monomial map is induced by a linear map on the set of monomials and its matrix is

$$M_1 := \begin{pmatrix} 1 + s\beta_1\alpha'_1 & s\beta_1\alpha'_2 & s\beta_1\alpha'_3 & \cdots & s\beta_1\alpha'_n \\ s\beta_2\alpha'_1 & 1 + s\beta_2\alpha'_2 & s\beta_2\alpha'_3 & \cdots & s\beta_2\alpha'_n \\ s\beta_3\alpha'_1 & s\beta_3\alpha'_2 & 1 + s\beta_3\alpha'_3 & \cdots & s\beta_3\alpha'_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s\beta_n\alpha'_1 & s\beta_n\alpha'_2 & s\beta_n\alpha'_3 & \cdots & 1 + s\beta_n\alpha'_n \end{pmatrix}$$

Set

$$M_2 := \begin{pmatrix} -s\beta_1\alpha'_1 & -s\beta_1\alpha'_2 & -s\beta_1\alpha'_3 & \cdots & -s\beta_1\alpha'_n \\ -s\beta_2\alpha'_1 & -s\beta_2\alpha'_2 & -s\beta_2\alpha'_3 & \cdots & -s\beta_2\alpha'_n \\ -s\beta_3\alpha'_1 & -s\beta_3\alpha'_2 & -s\beta_3\alpha'_3 & \cdots & -s\beta_3\alpha'_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s\beta_n\alpha'_1 & -s\beta_n\alpha'_2 & -s\beta_n\alpha'_3 & \cdots & -s\beta_n\alpha'_n \end{pmatrix}$$

and let $\chi(t)$ be the characteristic polynomial of M_2 . Then $\chi(1) = \det(M_1)$. If $\chi(1) = 0$, then the vector $\beta := (\beta_1, \dots, \beta_n)$ is an eigenvector of M_2 with eigenvalue 1 since the image of M_2 is generated by β . Thus $-s(\beta_1\alpha'_1 + \dots + \beta_n\alpha'_n) = 1$ which is not possible since $\beta_i \geq 0$ and $\alpha'_i > 0$ for any i . Thus $\det(M_1) \neq 0$ and M_1 is invertible. In particular $\sigma := M_1^{-1}(\mathbb{R}_{\geq 0}^n)$ is a rational strictly convex cone. Moreover, since $A \in \mathcal{V}_{\alpha, \delta}$, we have $\langle \alpha, \tau \rangle > 0$ for any $\tau \in \sigma$, $\tau \neq 0$. By Corollary 5.7 this proves the first assertion.

By Example 3.15 integral homogeneous elements with respect to ν_α are either finite over \mathbb{k} , either of the form $cx_1^{\frac{n_1}{q}} \dots x_n^{\frac{n_n}{q}}$ for some integers $n_1, \dots, n_n \in \mathbb{N}$, $q \in \mathbb{N}_{>0}$ such that $\sum_{j=1}^n \alpha_j n_j > 0$. Using Theorem 5.6 and since $\overline{\mathbb{K}}_{\nu_\alpha} = \mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_s]$ where the γ_i are homogeneous with respect to ν_α , we have the second inclusion by replacing σ by the rational strictly convex cone generated by σ and the (n_1, \dots, n_n) corresponding to the homogeneous elements $\gamma_1, \dots, \gamma_s$. □

We will consider a subring R of $\mathbb{k}[[\mathbf{x}]]$ that is an excellent Henselian local ring with maximal ideal \mathfrak{m}_R such that:

- (A) $\mathbb{k}[x_1, \dots, x_n]_{(\mathbf{x})} \subset R$,
- (B) $\mathfrak{m}_R = (\mathbf{x})R$ and $\widehat{R} = \mathbb{k}[[\mathbf{x}]]$,
- (C) if $p(\mathbf{x}) \in \mathbb{k}[\mathbf{x}]$ and $f(\mathbf{x}) \in R$, then $f(p(\mathbf{x})x_1, \dots, p(\mathbf{x})x_n) \in R$.

Remark 6.10. If \mathbb{k} is a field, the ring of algebraic power series $\mathbb{k}\langle \mathbf{x} \rangle$ is an excellent Henselian local ring satisfying Properties (A), (B) and (C). If \mathbb{k} is a valued field, then the field of convergent power series $\mathbb{k}\{\mathbf{x}\}$ does also.

Definition 6.11. Let $\alpha \in \mathbb{R}_{>0}^n$ and let δ be a (α) -homogeneous polynomial. Let

$$\mathcal{V}_{\alpha, \delta}^R := \left\{ A \in \widehat{V}_{\nu_\alpha} / \exists \Lambda \text{ a finitely generated semigroup of } \mathbb{R}_{\geq 0}, \forall i \in \Lambda \exists a_i \in \mathbb{k}[\mathbf{x}] \text{ } (\alpha)\text{-homogeneous,} \right.$$

$$\exists a, b \geq 0 \forall i \in \Lambda \exists m(i) \in \mathbb{N} \text{ s.t. } m(i) \leq ai + b, \nu_\alpha \left(\frac{a_i}{\delta^{m(i)}} \right) = i, A = \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}}$$

$$\text{and } \exists \varepsilon > 0 \forall q \in \mathbb{N} \forall \alpha' \in \text{Rel}(\alpha, q, \varepsilon) \exists s \in \mathbb{N} \text{ such that } \delta(\mathbf{x})^s \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}} (\mathbf{x} \delta(\mathbf{x})^{s\alpha'}) \in R \Big\}.$$

Remark 6.12. Let $\sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}} \in \mathcal{V}_{\alpha, \delta}^R$. Then there exists $\varepsilon > 0$ such that for any $q \in \mathbb{N}$ and any $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ there exists $s \in \mathbb{N}$ such that (by Equation (5)):

$$\delta(\mathbf{x})^s \sum_{i \in \Lambda} \frac{a_i}{\delta_i^{m(i)}} (x_1 \delta(\mathbf{x})^{\alpha'_1 s}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n s}) = \sum_{i \in \Lambda} a_i(\mathbf{x}) \delta(\mathbf{x})^{s(1+\nu_{\alpha'}(a_i(\mathbf{x}))-\nu_{\alpha'}(\delta(\mathbf{x}))m(i))-m(i)} \in R.$$

If $t \in \mathbb{N}$, then

$$\begin{aligned} & \delta(\mathbf{x})^t \sum_{i \in \Lambda} a_i(x_1 \delta(\mathbf{x})^{\alpha'_1 t}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n t}) \delta(x_1 \delta(\mathbf{x})^{\alpha'_1 t}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n t})^{s(1+\nu_{\alpha'}(a_i(\mathbf{x}))-\nu_{\alpha'}(\delta(\mathbf{x}))m(i))-m(i)} = \\ &= \sum_{i \in \Lambda} a_i(\mathbf{x}) \delta(\mathbf{x})^{t+t[\nu_{\alpha'}(a_i(\mathbf{x}))+\nu_{\alpha'}(\delta(\mathbf{x}))][s(1+\nu_{\alpha'}(a_i(\mathbf{x}))-\nu_{\alpha'}(\delta(\mathbf{x}))m(i))-m(i)]+s(1+\nu_{\alpha'}(a_i(\mathbf{x}))-\nu_{\alpha'}(\delta(\mathbf{x}))m(i))-m(i)} \\ &= \sum_{i \in \Lambda} a_i(\mathbf{x}) \delta(\mathbf{x})^{d_{\alpha'} t s p_{\alpha'}(i) + t p_{\alpha'}(i) + s p_{\alpha'}(i) - m(i) + d_{\alpha'} t s + t + s} \in R \end{aligned}$$

where

$$p_{\alpha'}(i) := \nu_{\alpha'}(a_i(\mathbf{x})) - \nu_{\alpha'}(\delta(\mathbf{x}))m(i) \text{ and } d_{\alpha'} := \nu_{\alpha'}(\delta(\mathbf{x}))$$

by Property (C). In particular

$$(6) \quad \delta^{d_{\alpha'} s t + s + t} \sum_{i \in \Lambda} a_i \delta^{(d_{\alpha'} s t + s + t) p_{\alpha'}(i) - m(i)} \in R \quad \forall t \in \mathbb{N}.$$

If $A = \sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}}$ and $B = \sum_{i \in \Lambda} \frac{b_i}{\delta^{n(i)}}$ $\in \mathcal{V}_{\alpha, \delta}^R$, then there exist $s_1, s_2 \in \mathbb{N}$ such that

$$\delta^{s_1} \sum_{i \in \Lambda} a_i \delta^{s_1 p_{\alpha', 1}(i) - m(i)} \in R \text{ and } \delta^{s_2} \sum_{i \in \Lambda} b_i \delta^{s_2 p_{\alpha', 2}(i) - n(i)} \in R$$

where

$$p_{\alpha', 1}(i) := \nu_{\alpha'}(a_i(\mathbf{x})) - \nu_{\alpha'}(\delta(\mathbf{x}))m(i) \text{ and } p_{\alpha', 2}(i) := \nu_{\alpha'}(b_i(\mathbf{x})) - \nu_{\alpha'}(\delta(\mathbf{x}))n(i).$$

Let p be a prime number and $k \in \mathbb{N}_{>0}$ such that p^k divides $d_{\alpha'} s_1 + 1$ and $d_{\alpha'} s_2 + 1$. Then $\gcd(p, d_{\alpha'}) = 1$ and p^k divides $d_{\alpha'} s_1 - d_{\alpha'} s_2$. Thus p^k divides $s_1 - s_2$. This proves that $\gcd(d_{\alpha'} s_1 + 1, d_{\alpha'} s_2 + 1)$ divides $s_1 - s_2$. Thus there exist $t_1 \in \mathbb{Z}^*$ and $t_2 \in \mathbb{Z}^*$ such that $(d_{\alpha'} s_1 + 1)t_1 - (d_{\alpha'} s_2 + 1)t_2 = s_2 - s_1$. If $t_1 t_2 < 0$, let say $t_1 > 0$ and $t_2 < 0$, then $(d_{\alpha'} s_1 + 1)t_1 - (d_{\alpha'} s_2 + 1)t_2 > s_1 + s_2 > |s_1 - s_2|$ which is not possible. Thus we have that $t_1 t_2 > 0$. If $t_1 < 0$ and $t_2 < 0$, we can replace t_1 (resp. t_2) by $t_1 + k(d_{\alpha'} s_2 + 1)$ (resp. by $t_2 + k(d_{\alpha'} s_1 + 1)$) for some positive integer k large enough. This will allow to assume that t_1 and t_2 are positive integer. Hence

$$\exists t_1, t_2 \in \mathbb{N}, d_{\alpha'} s_1 t_1 + s_1 + t_1 = d_{\alpha'} s_2 t_2 + s_2 + t_2.$$

This proves that there exists $s \in \mathbb{N}$ ($s = d_{\alpha'} s_1 t_1 + s_1 + t_1 = d_{\alpha'} s_2 t_2 + s_2 + t_2$) such that

$$\delta(\mathbf{x})^s \sum_{i \in \Lambda} \frac{a_i}{\delta_i^{m(i)}} (x_1 \delta(\mathbf{x})^{\alpha'_1 s}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n s}) \text{ and } \delta(\mathbf{x})^s \sum_{i \in \Lambda} \frac{b_i}{\delta_i^{n(i)}} (x_1 \delta(\mathbf{x})^{\alpha'_1 s}, \dots, x_n \delta(\mathbf{x})^{\alpha'_n s}) \in R.$$

Thus $A + B \in \mathcal{V}_{\alpha, \delta}^R$ and $AB \in \mathcal{V}_{\alpha, \delta}^R$. This proves that $\mathcal{V}_{\alpha, \delta}^R$ is a ring. Thus \mathcal{V}_α^R is also a ring and it is straightforward to check that it is a valuation ring.

Example 6.13. If $\alpha \in \mathbb{N}^n$ and $R = \mathbb{C}\{\mathbf{x}\}$ is the ring of convergent power series over \mathbb{C} , we claim that

$$\mathcal{V}_{\alpha, \delta}^{\mathbb{C}\{\mathbf{x}\}} = \left\{ \sum_{i \in \mathbb{N}} \frac{a_i}{\delta^{a(i+1)}} \mid \forall i \ a_i \in \mathbb{C}[\mathbf{x}] \text{ is } (\alpha)\text{-homogeneous, } \nu_\alpha \left(\frac{a_i}{\delta^{a(i+1)}} \right) = i, \right. \\ \left. a \in \mathbb{Z}_{\geq 0} \text{ and } \exists C, r > 0 \text{ such that } |a_i(z)| \leq Cr^i \|z\|_\alpha^{\nu_\alpha(a_i)} \ \forall z \in \mathbb{C}^n \right\}$$

where $\|z\|_\alpha := \max_{j=1, \dots, n} \left| z_j^{\frac{1}{\alpha_j}} \right|$ for any $z \in \mathbb{C}^n$.

First of all any element of \mathcal{V}_α is of the form $\sum_{i \in \mathbb{N}} \frac{a_i}{\delta^{m(i)}}$ where $\nu_\alpha \left(\frac{a_i}{\delta^{m(i)}} \right) = i$ and $m(i) \leq ai + b$

for some $a, b \in \mathbb{N}$. By multiplying the numerator and the denominator of $\frac{a_i}{\delta^{m(i)}}$ by $\delta^{ai+b-m(i)}$ and replacing a_i by $a_i \delta^{ai+b-m(i)}$, we may assume that $m(i) = ai + b$. If $a > b$, we may replace $\frac{a_i}{\delta^{ai+b}}$ by $\frac{a_i \delta^{a-b}}{\delta^{ai+a}}$, if $a < b$ we may replace $\frac{a_i}{\delta^{ai+b}}$ by $\frac{a_i \delta^{(b-a)i}}{\delta^{bi+b}}$. Thus any element of \mathcal{V}_α is of the form $\sum_{i \in \mathbb{N}} \frac{a_i}{\delta^{a(i+1)}}$ where $\nu_\alpha \left(\frac{a_i}{\delta^{a(i+1)}} \right) = i$ for all $i \in \mathbb{N}$. In this case

$$\nu_\alpha(a_i) = (a\nu_\alpha(\delta) + 1)i + a\nu_\alpha(\delta) \text{ for any } i \in \mathbb{N}.$$

Then we have (with $s = a$ in Lemma 6.6):

$$f(\mathbf{x}) := \delta(\mathbf{x})^a \sum_{i \in \Lambda} \frac{a_i}{\delta_i^{a(i+1)}} (x_1 \delta(\mathbf{x})^{\alpha_1 a}, \dots, x_n \delta(\mathbf{x})^{\alpha_n a}) = \sum_{i \in \Lambda} a_i(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]].$$

Thus $f \in \mathbb{C}\{x\}$ if and only if this power series is convergent on a neighborhood of the origin. This neighborhood may be chosen of the form:

$$B_\alpha(0, r) := \{z \in \mathbb{C}^n \mid |z_j| \leq r^{\alpha_j}, j = 1, \dots, n\}.$$

For any $z \in B_\alpha(0, r)$ set $t_j^{\alpha_j} = z_j$ for $j = 1, \dots, n$ and $b_i(t) = a_i(z)$ for any $i \in \mathbb{N}$. Then f is convergent on $B_\alpha(0, r)$ if and only $\sum_{i \in \mathbb{N}} b_i(t)$ is convergent on $B(0, r) := \{t \in \mathbb{C}^n \mid |t_j| \leq$

$r, j = 1, \dots, n\}$. But this series is convergent if and only if there exist $c \geq 0$ and $\rho < 1$ such that $|b_i(t)| \leq c\rho^i$ for all $i \in \mathbb{N}$ and all $t \in B(0, r)$. Since $b_i(t)$ is a homogeneous polynomial of degree $\nu_\alpha(a_i) = (ad + 1)i + ad$ where $d := \nu_\alpha(\delta)$, we have

$$\sup_{|t_j| \leq r, j=1 \dots n} |b_i(t)| = r^{(ad+1)i+ad} \sup_{|t_j| \leq 1, j=1 \dots n} |b_i(t)|$$

we see that f is convergent if and only if there exist $C \geq 0$ and $R > 0$ such that

$$\sup_{|z_j| \leq 1, j=1 \dots n} |a_i(z)| = \sup_{|t_j| \leq 1, j=1 \dots n} |b_i(t)| \leq CR^i.$$

In this case we have

$$(7) \quad |a_i(z)| = |b_i(t)| \leq \max_{j=1, \dots, n} |t_j| \sup_{|t_j| \leq 1, j=1, \dots, n} |b_i(t)| \leq CR^i \|z\|_\alpha^{\nu_\alpha(a_i)} \quad \forall z \in \mathbb{C}^n.$$

This proves the claim.

We have the following analogue of Theorem 5.6 in the Henselian case:

Theorem 6.14. Let \mathbb{k} be a field of characteristic zero and let R be a subring of $\mathbb{k}[[\mathbf{x}]]$ that is an excellent Henselian local ring satisfying Properties (A), (B) and (C). Let $\alpha \in \mathbb{R}_{\geq 0}^n$ and let us set $N = \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n)$.

Let $P(Z) \in \mathcal{V}_\alpha^R[\langle \gamma_1, \dots, \gamma_s \rangle][Z]$ be a distinguished polynomial of degree d where the γ_i 's are

homogeneous elements with respect to ν_α . Then $P(Z)$ has all its roots in $\mathcal{V}_\alpha^R[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ where $\gamma'_1, \dots, \gamma'_N$ are integral homogeneous elements with respect to ν_α .

Proof. Let $P(Z) = Z^d + A_1 Z^{d-1} + \dots + A_d$ with $A_j \in \mathcal{V}_\alpha^R[\langle \gamma_1, \dots, \gamma_s \rangle]$ for $1 \leq j \leq d$ and let $z \in \mathcal{V}_{\sigma, \delta}[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ be a root of $P(Z)$. Let $\varepsilon > 0$, $q \in \mathbb{N}$, $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ and s satisfying Lemma 6.6. Let us denote by φ the following ring morphism defined on $\mathcal{V}_{\alpha, \delta}$:

$$A = \sum_i \frac{a_i}{\delta^{m(i)}} \mapsto \varphi(A) := \sum_i \frac{a_i(\delta^{\alpha'_1 s}(\mathbf{x})x_1, \dots, \delta^{\alpha'_n s}(\mathbf{x}))}{\delta^{m(i)}(\delta^{\alpha'_1 s}(\mathbf{x}), \dots, \delta^{\alpha'_n s}(\mathbf{x}))} = \sum_i a_i \delta^{s(\nu_{\alpha'}(a_i(\mathbf{x})) - \nu_{\alpha'}(\delta(\mathbf{x}))m(i)) - m(i)}.$$

We have the following lemma:

Lemma 6.15. *Let γ'_i be a homogeneous element with respect to ν_α , $1 \leq i \leq N$. Then there exist homogeneous element γ''_i with respect to $\nu_{\alpha'}$, $1 \leq i \leq N$, such that, for any finite number of elements $A_{i_1, \dots, i_N} \in \mathcal{V}_{\alpha, \delta}$,*

$$\varphi \left(\sum_{i_1, \dots, i_N} A_{i_1, \dots, i_N} \gamma_1'^{i_1} \dots \gamma_N'^{i_N} \right) := \sum_{i_1, \dots, i_N} \varphi(A_{i_1, \dots, i_N}) \gamma_1''^{i_1} \dots \gamma_N''^{i_N}$$

defines an extension of φ from $\mathcal{V}_{\alpha, \delta}[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ to $\mathcal{V}_{\alpha', \delta}[\langle \gamma''_1, \dots, \gamma''_N \rangle]$.

Proof of Lemma 6.15. Let us assume that γ'_i is a homogeneous element of degree e_i with respect to ν_α . Let $Q_i(Z) := g_{i,0}(\mathbf{x})Z^{q_i} + g_{i,1}(\mathbf{x})Z^{q_i-1} + \dots + g_{i,q_i}(\mathbf{x})$ be a polynomial such that $Q_i(\gamma'_i) = 0$ and such that $g_{i,j}(\mathbf{x})$ is a (α) -homogeneous polynomial of degree $d_i + j e_i$.

Then $g_{i,j}(\mathbf{x})$ is a (α') -homogeneous polynomial of degree $d'_i + j e'_i$ for some constants d'_i and e'_i . Indeed, if a, b and c are (α) -homogeneous polynomials and $\nu_\alpha(a) - \nu_\alpha(b) = \nu_\alpha(b) - \nu_\alpha(c)$, then ac and b^2 are two (α) -homogeneous polynomials of same degree, i.e. $ac - b^2$ is (α) -homogeneous. Then, by Lemma 6.4, $ac - b^2$ is (α') -homogeneous, thus $\nu_{\alpha'}(a) - \nu_{\alpha'}(b) = \nu_{\alpha'}(b) - \nu_{\alpha'}(c)$.

Let $\overline{Q}_i(Z) := g_{i,0}(\mathbf{x})Z^{q_i} + g_{i,1}(\mathbf{x})\delta(\mathbf{x})^{se'_i}Z^{q_i-1} + \dots + g_{i,q_i}(\mathbf{x})\delta(\mathbf{x})^{se'_i q_i}$ and let γ''_i be a root of $\overline{Q}_i(Z)$. Then γ''_i is a homogeneous element of degree $e'_i(1 + \nu_{\alpha'}(\delta(\mathbf{x}))s)$ with respect to $\nu_{\alpha'}$. Then it is straightforward to check that $\varphi(\sum A_{i_1, \dots, i_N} \gamma_1'^{i_1} \dots \gamma_N'^{i_N}) = \sum \varphi(A_{i_1, \dots, i_N}) \gamma_1''^{i_1} \dots \gamma_N''^{i_N}$ defines an extension of φ from $\mathcal{V}_{\alpha, \delta}[\langle \gamma'_1, \dots, \gamma'_N \rangle]$ to $\mathcal{V}_{\alpha', \delta}[\langle \gamma''_1, \dots, \gamma''_N \rangle]$. \square

By Lemma 6.15, Remark 6.12 and Property (C), $\delta^{js}\varphi(A_j) \in R[\gamma''_1, \dots, \gamma''_N]$ for $1 \leq j \leq d$. By Equation (6) of Remark 6.12 and Property (C), we may even assume that $\delta^s\varphi(z) \in \mathbb{k}[\mathbf{x}][\gamma''_1, \dots, \gamma''_N]$ by taking s large enough. Thus $z' := \delta^s\varphi(z) \in \mathbb{k}[\mathbf{x}][\gamma''_1, \dots, \gamma''_N]$ is a root of $\overline{P}(Z) := Z^d + \delta^s\varphi(A_1)Z^{d-1} + \dots + \delta^{ds}\varphi(A_d) \in R[Z]$. Let us write

$$z' := \sum_{i_1, \dots, i_N} z'_{i_1, \dots, i_N} \gamma_1''^{i_1} \dots \gamma_N''^{i_N}$$

with $z'_{i_1, \dots, i_N} \in \mathbb{k}[\mathbf{x}]$ for any i_1, \dots, i_N . Let $Z := \sum_{i_1, \dots, i_N} Z_{i_1, \dots, i_N} \gamma_1''^{i_1} \dots \gamma_N''^{i_N}$ where Z_{i_1, \dots, i_N} are new variables. Solving $P(Z) = 0$ is equivalent to solve a finite system (\mathcal{S}) of polynomial equations in the variables Z_{i_1, \dots, i_N} with coefficients in R , just by replacing Z by $\sum_{i_1, \dots, i_N} Z_{i_1, \dots, i_N} \gamma_1''^{i_1} \dots \gamma_N''^{i_N}$ and replacing the high powers of the γ'_i 's by smaller ones using the division by the $Q_i(Z_i)$'s. By Artin approximation theorem (cf. [Po], [Sp2]), the set of solutions of (\mathcal{S}) in R is dense in the set of solutions in $\mathbb{k}[\mathbf{x}]$, but since $P(Z) = 0$ has a finite number of solutions, then (\mathcal{S}) has a finite number of solutions and they are in R . Thus $z'_{i_1, \dots, i_N} \in R$ for all i_1, \dots, i_N , hence $z' \in R[\gamma''_1, \dots, \gamma''_N]$. This proves that $z \in \mathcal{V}_{\alpha, \delta}^R[\langle \gamma'_1, \dots, \gamma'_N \rangle]$. \square

7. A GENERALIZATION OF ABHYANKAR-JUNG THEOREM

Definition 7.1. Let $\alpha \in \mathbb{N}^n$ and let $\theta \in \mathbb{C}[\mathbf{x}]$ be a (α) -homogeneous polynomial. Let $a > 0$, $C > 0$ and $\eta > 0$. Set :

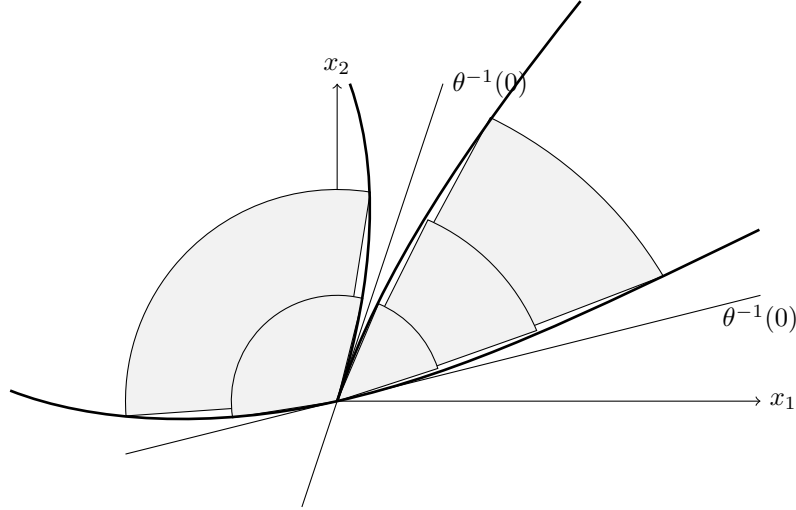
$$\mathcal{D}_{\theta, C, a, \eta} := \left(\bigcup_{\substack{K > 0, \varepsilon > 0 \\ \varepsilon < K^a C}} C_{K, \varepsilon} \right) \cap B(0, \eta)$$

with

$$C_{K, \varepsilon} := \{x \in \mathbb{C}^n / d_\alpha(x, \theta^{-1}(0)) > K \|x\|_\alpha \text{ and } \|x\|_\alpha < \varepsilon\}$$

where $\|\cdot\|_\alpha$ is defined in Example 6.13 and d_α is defined as follows: for any $x, y \in \mathbb{C}^n$ let us denote by $x_i^{\frac{1}{\alpha_i}}$ (resp. $y_i^{\frac{1}{\alpha_i}}$) a complex α_i -th root of x_i (resp. y_i) and let \mathbb{U}_i be the set of α_i -roots of unity. Then we define $d_\alpha(x, y) := \max_i \inf_{\xi \in \mathbb{U}_i} \left| x_i^{\frac{1}{\alpha_i}} - \xi y_i^{\frac{1}{\alpha_i}} \right|$ and $d_\alpha(x, \theta^{-1}(0)) := \inf_{x' \in \theta^{-1}(0)} d_\alpha(x, x')$.

Then $\mathcal{D}_{\theta, C, a, \eta}$ is the complement of a hornshaped neighborhood of $\{\theta = 0\}$ as we can see on the following picture (here $\alpha = (1, \dots, 1)$):



Lemma 7.2. Let $a \in \mathbb{N}^n$ and $A \in \mathcal{V}_{\alpha, \theta}^{\mathbb{C}\{\mathbf{x}\}}$. Then there exists $a > 0$ and $C > 0$ such that A is analytic on $\mathcal{D}_{\theta, C, a, \eta}$ for any $\eta > 0$.

Proof. If $\nu_\alpha(a_i) = d_i$ there exist $C > 0$ and $r > 0$ such that

$$(8) \quad |a_i(x)| \leq C r^i \|x\|_\alpha^{d_i} \quad \forall x \in \mathbb{C}^n$$

by Theorem 6.14, Example 6.13 and Inequality (7) of Example 6.13. On the other hand we claim that there exists a constant $C' > 0$ such that

$$(9) \quad |\theta(x)| \geq C' d_\alpha(x, \theta^{-1}(0))^{\nu_\alpha(\theta)} \quad \forall x \in \mathbb{C}^n$$

where $d_\alpha(x, \theta^{-1}(0)) := \inf_{x' \in \theta^{-1}(0)} d_\alpha(x, x')$.

Indeed if we embed $\mathbb{C}\{\mathbf{x}\}$ in $\mathbb{C}\{\mathbf{y}\}$ by sending x_i onto $y_i^{\alpha_i}$, we have $\theta(\mathbf{x}) = \theta(y_1^{\alpha_1}, \dots, y_n^{\alpha_n}) =$

$\tau(y_1, \dots, y_n)$ and τ is a homogeneous polynomial of degree $d = \nu_\alpha(\theta)$. After a linear change of coordinates, we may assume that τ is a monic polynomial in y_n of degree d multiplied by a constant. Then, for all $y_1, \dots, y_n \in \mathbb{C}^n$, we have

$$|\tau(y_1, \dots, y_n)| = C' \left| \prod_{i=1}^{d'} (y_n - \varphi_i(y_1, \dots, y_{n-1})) \right|$$

where φ_i is a homogeneous function which is locally analytic outside the discriminant locus of τ , for some constant C' . Thus

$$|\tau(y_1, \dots, y_n)| \geq C' \min_i |y_n - \varphi_i(y_1, \dots, y_{n-1})|^d \geq C' \inf_{y' \in \tau^{-1}(0)} \max_k |y_k - y'_k|^d = C' d(y, \tau^{-1}(0))^d$$

since $(y_1, \dots, y_{n-1}, \varphi_i(y_1, \dots, y_{n-1})) \in \tau^{-1}(0)$ for any i . This proves (9).

Hence we have

$$\left| \frac{a_i(x)}{\theta^{m(i)}(x)} \right| \leq \frac{C}{C'^{m(i)}} \frac{\|x\|_\alpha^{d_i}}{d_\alpha(x, \theta^{-1}(0))^{dm(i)}} r^i \leq \frac{C\varepsilon^i}{C'^{m(i)} K^{dm(i)}} \quad \forall x \in C_{K,\varepsilon}.$$

Let $i \mapsto ai + b$ be a bounding function for A . Then if $\varepsilon < K^{ad} C'^a$, A defines an analytic function on the domain $C_{K,\varepsilon}$. Thus A defines an analytic function on the domain $\mathcal{D}_{\theta, C'^a, ad, \eta}$ for any $\eta > 0$. \square

This following proposition has been proven by Tougeron in the case $\alpha = (1, \dots, 1)$ (see Proposition 2.8 [To]):

Proposition 7.3. *Let $\alpha \in \mathbb{N}^n$ and let $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$ be a monic polynomial whose discriminant is equal to δu where $\delta \in \mathbb{C}\{\mathbf{x}\}$ is (α) -homogeneous and $u \in \mathbb{C}\{\mathbf{x}\}$ is invertible. If $P(Z) = P_1(Z) \dots P_r(Z)$ where $P_i(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$ is irreducible in $\mathbb{C}\{\mathbf{x}\}[Z]$, then $P_i(Z)$ is irreducible in $\mathcal{V}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[Z]$.*

Proof. Let $Q(Z)$ be an irreducible factor of $P(Z)$ in $\mathcal{V}_\alpha[Z]$. From Theorem 5.6, there exists $\theta \in \mathbb{C}[x]$ a (α) -homogeneous polynomial such that the coefficients of $Q(Z)$ are in $\mathcal{V}_{\nu_\alpha, \theta}$. Let $A := \sum_{i \in \Lambda} \frac{a_i}{\theta^{m(i)}}$ be such a coefficient where Λ is a finitely generated sub-semigroup of $\mathbb{R}_{\geq 0}$. Since $\mathcal{V}_{\nu_\alpha, \theta} \subset \mathcal{V}_{\nu_\alpha, \theta\delta}$, we may assume that δ divides θ , thus $\Delta^{-1}(0) \cap B(0, \varepsilon) \subset \theta^{-1}(0) \cap B(0, \varepsilon)$ for $\varepsilon > 0$ small enough ($B(0, \varepsilon)$ is the open ball centered in 0 and of radius ε). Let $\eta > 0$ small enough such that

$$\Delta^{-1}(0) \cap B(0, \eta) \subset \theta^{-1}(0) \cap B(0, \eta)$$

and the roots are locally analytic on the domain $D_{\theta, \eta} := B(0, \eta) \setminus \theta^{-1}(0) \subset B(0, \eta) \setminus \Delta^{-1}(0)$. Since A is a polynomial depending on the roots of $P(Z)$, then it is locally analytic on $D_{\theta, \eta}$. On the other hand, by Lemma 7.2 A defines an analytic function on the domain $\mathcal{D}_{\theta, C'^a, a, \eta}$. Thus by Lemma 7.4 A is global analytic on $D_{\theta, \eta}$. Since the roots of $P(Z)$ are bounded near the origin, A is bounded near the origin, thus A extends to an analytic function near the origin. This proves that A is analytic on a neighborhood of the origin and $Q(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$. \square

Lemma 7.4. *Set $C > 0$, $a > 0$ and $\eta > 0$ and let $\theta \in \mathbb{C}[\mathbf{x}]$ be a (α) -homogeneous polynomial. Let $A : D_{\theta, \eta} \rightarrow \mathbb{C}$ be a multivalued function. Let us assume that A is analytic on $\mathcal{D}_{\theta, C, a, \eta}$ and locally analytic on $D_{\theta, \eta}$. Then A is analytic on $D_{\theta, \eta}$.*

Proof. Since A is locally analytic on $D_{\theta,\eta}$, then A extends to an analytic function on a small neighborhood of every path in $D_{\theta,\eta}$. If A is not global analytic on $D_{\theta,\eta}$, then there exists a loop based at a point p of $D_{\theta,\eta}$, denoted by $\varphi : [0, 1] \rightarrow D_{\theta,\eta}$ with $\varphi(0) = \varphi(1) = p$, such that A extends to an analytic function on a neighborhood of φ but $A \circ \varphi(0) \neq A \circ \varphi(1)$. Let us write $\varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$ and let us define $\Phi : [0, 1] \times S \rightarrow \mathbb{C}^n$ by

$$\Phi(t, s) := (s^{\alpha_1} \varphi_1(t), \dots, s^{\alpha_n} \varphi_n(t))$$

where $S := \{z \in \mathbb{C} \mid |z| \leq 1, \Re(z) > 0\}$.

Then we have $\delta(\Phi(t, s)) = s^{\nu_\alpha(\delta)} \delta(\varphi(t)) \neq 0$ for any $(t, s) \in [0, 1] \times S$ since $\text{Im}(\varphi) \subset D_{\theta,\eta}$ and $s \neq 0$. Thus the image of Φ is included in $D_{\theta,\eta}$. Moreover, for any $t \in [0, 1]$, let $\Phi_t : S \rightarrow D_{\theta,\eta}$ be the function defined by $\Phi_t(s) := \Phi(t, s)$. Its image is simply connected since S is simply connected and Φ_t is analytic. Thus $A \circ \Phi_t$, which is locally analytic, extends to an analytic function on S by the Monodromy Theorem.

Let us denote by h the holomorphic function on S defined by $h(s) := A \circ \Phi(0, s) - A \circ \Phi(1, s)$ for any $s \in S$.

For any $s \in S$ and any $t \in [0, 1]$ we have

$$\|\varphi(t, s)\|_\alpha = |s| \|\varphi(t)\|_\alpha \quad \text{and} \quad d_\alpha(\varphi(t, s), \theta^{-1}(0)) = |s| d_\alpha(\varphi(t), \theta^{-1}(0)).$$

Let us set

$$K := \frac{1}{2} \min_{t \in [0, 1]} \frac{d_\alpha(\varphi(t, s), \theta^{-1}(0))}{\|\varphi(t, s)\|_\alpha} = \frac{1}{2} \min_{t \in [0, 1]} \frac{d_\alpha(\varphi(t), \theta^{-1}(0))}{\|\varphi(t)\|_\alpha} > 0.$$

Thus for any s belonging to the domain $S \cap \{|s| < K^a C\}$, we have $\varphi(t, s) \in \mathcal{D}_{\theta, C, a, \eta}$. Since $\varphi(t, s) \in \mathcal{D}_{\theta, C, a, \eta}$ and A is analytic on $\mathcal{D}_{\theta, C, a, \eta}$, then $A \circ \varphi(0, s) = A \circ \varphi(1, s)$, thus $h(s) = 0$ on $S \cap \{|s| < K^a C\}$. Since h is holomorphic on the connected domain S , then $h \equiv 0$ on S . This contradicts the assumption. Hence A is global analytic on $D_{\theta,\eta}$. \square

Then we can extend Proposition 7.3 to the formal setting over any field of characteristic zero:

Theorem 7.5. *Let \mathbb{k} be a field of characteristic zero and $\alpha \in \mathbb{R}_{>0}^n$. Let $P(Z) \in \mathcal{F}_n[Z]$ be a monic polynomial whose discriminant is equal to δu where $\delta \in \mathbb{k}[\mathbf{x}]$ is (α) -homogeneous and $u \in \mathcal{F}_n$ is a unit. If $P(Z)$ factors as $P(Z) = P_1(Z) \dots P_s(Z)$ where $P_i(Z)$ is an irreducible monic polynomial of $\mathcal{F}_n[Z]$, then $P_i(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$.*

Proof. Let us prove this theorem when $P(Z) \in \mathbb{C}[\mathbf{x}][Z]$. If $\alpha \in \mathbb{N}^n$, this is exactly Proposition 7.3. If $\alpha \notin \mathbb{N}^n$, then by Lemma 6.7, any decomposition $P(Z) = Q_1(Z) \dots Q_r(Z)$ in $\mathcal{V}_\alpha[Z]$ gives a decomposition in $\mathcal{V}_{\alpha'}[Z]$ for $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ for ε small enough. Then we conclude with Proposition 7.3.

Now let us consider the general case. Let $P(Z) = Z^d + a_{d-1}(\mathbf{x})Z^{d-1} + \dots + a_0(\mathbf{x})$ be such a polynomial with $a_k(\mathbf{x}) \in \mathcal{F}_n$ for $0 \leq k \leq d-1$. Since $P(Z)$ is defined over a field extension of \mathbb{Q} generated by countably many elements and since such a field extension embeds in \mathbb{C} , we may assume that \mathbb{C} is a field extension of \mathbb{k} and $P(Z) \in \mathbb{C}[\mathbf{x}]$.

The discriminant of $P(Z)$ is a polynomial depending on $a_0(\mathbf{x}), \dots, a_{d-1}(\mathbf{x})$ that we denote by $D(a_0(\mathbf{x}), \dots, a_{d-1}(\mathbf{x}))$. Let

$$R(A_0, \dots, A_{d-1}, U) := D(A_0, \dots, A_{d-1}) - \delta(\mathbf{x})U \in \mathbb{C}[\mathbf{x}][A_0, \dots, A_{d-1}, U].$$

Then $R(a_0(\mathbf{x}), \dots, a_{d-1}(\mathbf{x}), u(\mathbf{x})) = 0$.

On the other hand, saying that $P(Z)$ factors as $P = P_1 \dots P_s$ is equivalent to

$$\exists b_1(\mathbf{x}), \dots, b_r(\mathbf{x}) \text{ such that } a_i(\mathbf{x}) = R_i(b_1(\mathbf{x}), \dots, b_r(\mathbf{x})) \quad \forall i$$

for some polynomials $R_i(B_1, \dots, B_r) \in \mathbb{Q}[B_1, \dots, B_r]$, $0 \leq i \leq d-1$ (these R_i are the coefficients of Z^i in $P(Z)$ and the b_j 's are the coefficients of the $P_k(Z)$'s).

By Artin Approximation Theorem [Art], for any integer $c > 0$, there exists convergent power series $\bar{a}_{0,c}(\mathbf{x}), \dots, \bar{a}_{d-1,c}(\mathbf{x}), \bar{u}_c(\mathbf{x}), \bar{b}_{1,c}(\mathbf{x}), \dots, \bar{b}_{r,c}(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}$ such that

$$(10) \quad R(\bar{a}_{0,c}(\mathbf{x}), \dots, \bar{a}_{d-1,c}(\mathbf{x}), \bar{u}_c(\mathbf{x})) = 0,$$

$$(11) \quad \bar{a}_{i,c}(\mathbf{x}) - R_i(\bar{b}_{1,c}(\mathbf{x}), \dots, \bar{b}_{r,c}(\mathbf{x})) = 0 \text{ for } 0 \leq i \leq d-1$$

and

$$\bar{a}_{k,c}(\mathbf{x}) - a_k(\mathbf{x}), \bar{u}_c(\mathbf{x}) - u(\mathbf{x}), \bar{b}_{l,c}(\mathbf{x}) - b_l(\mathbf{x}) \in (\mathbf{x})^c \text{ for } 0 \leq k \leq d, 1 \leq l \leq r.$$

Let $P_{(c)}(Z) := Z^d + \bar{a}_{d-1,c}(\mathbf{x})Z^{d-1} + \dots + \bar{a}_{0,c}(\mathbf{x})$. Then $P_{(c)}(Z)$ factors as

$$P_{(c)}(Z) = P_{1,(c)}(Z) \dots P_{s,(c)}(Z)$$

in $\mathbb{C}\{\mathbf{x}\}[Z]$ because of Equation (11), and $P_{i,(c)}(Z) - P_i(Z) \in (\mathbf{x})^c \mathcal{F}_n[Z]$ for $1 \leq i \leq s$. Moreover the discriminant of $P_{(c)}(Z)$ is of the form $\delta(\mathbf{x})u_{(c)}$ where $u_{(c)}$ is a unit in $\mathbb{C}\{\mathbf{x}\}$ if c is large enough by Equation (10). Since $P_i(Z)$ is irreducible in $\mathcal{F}_n[Z]$, then $P_{i,(c)}(Z)$ is irreducible in $\mathcal{F}_n[Z]$ for all i for c large enough (let us say for $c \geq c_0$). Moreover we can remark that $\nu_\alpha(a) \geq \min_i \{\alpha_i\} \text{ord}(a)$ for any $a \in \mathcal{F}_n$, thus $\nu_\alpha(\bar{a}_{k,c}(\mathbf{x}) - a_k(\mathbf{x}))$ increases linearly with c .

Let $c \geq c_0$ and let us assume that $P_{i,(c)}(Z)$ is not irreducible in $\mathcal{V}_\alpha[Z]$, let us say $P_{i,(c)}(Z) = P_{i,(c),1}(Z)P_{i,(c),2}(Z)$ with $P_{i,(c),1}(Z), P_{i,(c),2}(Z) \in \mathcal{V}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[Z]$ with $\deg_Z(P_{i,(c),k}(Z)) > 0$ for $k = 1, 2$. By Proposition 7.3 we see that $P_{i,(c),1}(Z), P_{i,(c),2}(Z) \in \mathbb{C}\{x\}[Z]$, and by Proposition 7.7 $P_{i,(c),1}(Z), P_{i,(c),2}(Z) \in \mathbb{L}\{x\}[Z]$ where \mathbb{L} is a subfield of \mathbb{C} which is finite over \mathbb{k} . Thus $\mathbb{L} = \mathbb{k}[\gamma]$ by the Primitive Element Theorem where γ is a homogeneous element of degree 0 with respect to ν_α by Example 3.16. But we have $\mathcal{V}_\alpha \cap \mathbb{k}[\gamma] = \mathbb{k}$. Thus $P_{i,(c),1}(Z), P_{i,(c),2}(Z) \in \mathbb{k}\{\mathbf{x}\}[Z] \subset \mathbb{k}[[\mathbf{x}]] [Z]$ which contradicts the assumption that $P_{i,(c)}$ is irreducible in $\mathcal{F}_n[Z]$. Thus $P_{i,(c)}(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. Hence, by Corollary 4.11, $P_i(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$ since $\nu_\alpha(\bar{a}_{k,c}(\mathbf{x}) - a_k(\mathbf{x}))$ increases linearly with c . \square

Remark 7.6. Let $P(Z) \in \mathcal{F}_n[Z]$ be a monic irreducible polynomial whose discriminant is equal to δu where $\delta \in \mathbb{k}[\mathbf{x}]$ is (α) -homogeneous and $u \in \mathcal{F}_n$ is a unit. Let \mathbb{L} be a normal closure of $\frac{\mathbb{K}_n[Z]}{(P(Z))}$ in $\overline{\mathbb{K}_{\nu_\alpha}^{\text{alg}}}$. For any Galois field extension $\mathbb{k}_1 \rightarrow \mathbb{k}_2$ let us denote by $\text{Gal}(\mathbb{k}_2/\mathbb{k}_1)$ its Galois group. Then, by Theorem 6 [McC], $\mathbb{L}.\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ is a Galois extension of $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ and

$$\text{Gal}(\mathbb{L}/\mathbb{K}_n) \simeq \text{Gal}(\mathbb{L}.\mathbb{K}_{\nu_\alpha}^{\text{alg}}/\mathbb{K}_{\nu_\alpha}^{\text{alg}}).$$

Moreover this isomorphism is defined as follows: if u_1, \dots, u_k is a \mathbb{K}_n -basis of \mathbb{L} , then it is a $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ -basis of $\mathbb{L}.\mathbb{K}_{\nu_\alpha}^{\text{alg}}$. Thus if $\sigma \in \text{Gal}(\mathbb{L}/\mathbb{K}_n)$, its image $\sigma' \in \text{Gal}(\mathbb{L}.\mathbb{K}_{\nu_\alpha}^{\text{alg}}/\mathbb{K}_{\nu_\alpha}^{\text{alg}})$ is defined by

$$\sigma'(A_1 u_1 + \dots + A_k u_k) := A_1 \sigma(u_1) + \dots + A_k \sigma(u_k)$$

for any $A_1, \dots, A_k \in \mathbb{K}_{\nu_\alpha}$.

On the other hand $\mathbb{L}.\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ is isomorphic to $\mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_N]$ for some integral homogeneous elements $\gamma_1, \dots, \gamma_N$ with respect to ν_α .

We claim that $\mathbb{K}_n \rightarrow \mathbb{K}_n[\gamma_1, \dots, \gamma_N]$ is a Galois field extension. Indeed, let $Q(Z) \in \mathbb{K}_n[Z]$ be an irreducible monic polynomial having a root z in $\mathbb{K}_n[\gamma_1, \dots, \gamma_N]$. Let $Q_1(Z) \in \mathbb{K}_{\nu_\alpha}^{\text{alg}}[Z]$ be a monic polynomial dividing $Q(Z)$ in $\mathbb{K}_{\nu_\alpha}^{\text{alg}}[Z]$ such that $Q_1(z) = 0$. Then the others roots of $Q_1(Z)$ are in $\mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_N]$ since $\mathbb{K}_{\nu_\alpha}^{\text{alg}} \rightarrow \mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_N]$ is a normal field extension. Thus these roots are conjugated under $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ -automorphisms of $\mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_N]$. But the

conjugates of γ_k under such automorphisms are homogeneous elements with respect to ν_α , since the minimal polynomial of γ_k over $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ is (ν_α, d) -homogeneous for some $d \in \mathbb{R}_{\geq 0}$. In particular the conjugates of γ_k under $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ -automorphisms of $\mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_N]$ are linear combinations of $\gamma_1, \dots, \gamma_N$ with coefficients in \mathbb{K}_n . Thus the coefficients of $Q_1(Z)$ are in $\mathbb{K}_{\nu_\alpha}^{\text{alg}} \cap \mathbb{K}_n[\gamma_1, \dots, \gamma_N] = \mathbb{K}_n$ and the other roots of $Q_1(Z)$ are in $\mathbb{K}_n[\gamma_1, \dots, \gamma_N]$. But since $Q(Z)$ is irreducible in $\mathbb{K}_n[Z]$, then $Q_1(Z) = Q(Z)$. Hence $Q(Z)$ splits in $\mathbb{K}_n[\gamma_1, \dots, \gamma_N][Z]$. This shows that $\mathbb{K}_n \rightarrow \mathbb{K}_n[\gamma_1, \dots, \gamma_N]$ is a normal field extension, thus a Galois field extension since $\text{char}(\mathbb{k}) = 0$.

Thus, still by Theorem 6 [McC], we have

$$\text{Gal}(\mathbb{L} \cdot \mathbb{K}_{\nu_\alpha}^{\text{alg}} / \mathbb{K}_{\nu_\alpha}^{\text{alg}}) \simeq \text{Gal}(\mathbb{K}_{\nu_\alpha}^{\text{alg}}[\gamma_1, \dots, \gamma_N] / \mathbb{K}_{\nu_\alpha}^{\text{alg}}) \simeq \text{Gal}(\mathbb{K}_n[\gamma_1, \dots, \gamma_N] / \mathbb{K}_n)$$

and these isomorphisms are defined in a similar way as the one from $\text{Gal}(\mathbb{L} / \mathbb{K}_n)$ to $\text{Gal}(\mathbb{L} \cdot \mathbb{K}_{\nu_\alpha}^{\text{alg}} / \mathbb{K}_{\nu_\alpha}^{\text{alg}})$. If $\alpha \in \mathbb{N}^n$ (we can always assume this by Lemma 6.4), then N may be chosen equal to 1 by Proposition 3.25, and we see that $\mathbb{K}_n[\gamma] \simeq \frac{\mathbb{K}_n[Z]}{(Q(Z))}$ where $Q(Z)$ is the minimal polynomial of γ over \mathbb{K}_n . Thus the Galois group of $P(Z)$ is isomorphic to the Galois group of one weighted homogeneous polynomial and this isomorphism is described as above.

The following proposition is a particular case of a result of S. Cutkosky and O. Kashcheyeva [CK] (see also Proposition 1 [AM]):

Proposition 7.7. *Let $\mathbb{k} \rightarrow \mathbb{k}'$ be a field extension. Let $f \in \mathbb{k}'[[\mathbf{x}]]$ be algebraic over $\mathbb{k}[[\mathbf{x}]]$ and let \mathbb{L} be the field extension of \mathbb{k} generated by all the coefficients of f . Then $\mathbb{k} \rightarrow \mathbb{L}$ is a finite field extension.*

Proof. Let $\alpha \in \mathbb{R}_{>0}^n$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$. By Theorem 5.6, $f \in \mathcal{V}_\alpha[\langle \gamma_1, \dots, \gamma_n \rangle]$ for some homogeneous element $\gamma_1, \dots, \gamma_n$ with respect to α . By Example 3.13 these homogeneous elements can be written as $\gamma_i = c_i \mathbf{x}^{\beta_i}$ where c_i is algebraic over \mathbb{k} and $\beta_i \in \mathbb{Q}^n$ for $1 \leq i \leq n$.

By expanding f either as a formal power series of $\mathbb{k}'[[\mathbf{x}]]$, $f = \sum_i b_i(\mathbf{x})$ where $b_i(\mathbf{x}) \in \mathbb{k}'[[\mathbf{x}]]$ is a (α) -homogeneous polynomial for any i , either as an element of $\mathcal{V}_\alpha[\langle \gamma_1, \dots, \gamma_n \rangle]$, $f = \sum_i \frac{a_i(\mathbf{x})}{\delta(\mathbf{x})^{m(i)}} \gamma_1^{k_1(i)} \dots \gamma_n^{k_n(i)}$, and by identifying the homogeneous terms of same valuation (which are monomials by Example 3.13), we obtain a countable number of relations as follows

$$b(\mathbf{x})\delta^m(\mathbf{x}) = \sum a_{n_1, \dots, n_s}(\mathbf{x})\gamma_1^{n_1} \dots \gamma_n^{n_n}$$

where $b(\mathbf{x})$ (corresponding to the $b_i(\mathbf{x})$) and $a_{n_1, \dots, n_s}(\mathbf{x})$ (corresponding to the $a_i(\mathbf{x})$) are monomials, $b(\mathbf{x}) \in \mathbb{k}'[[\mathbf{x}]]$, $a_{n_1, \dots, n_s}(\mathbf{x}) \in \mathbb{k}[[\mathbf{x}]]$, $m \in \mathbb{N}$, and the sum is finite. By dividing this equality by \mathbf{x}^β for β well chosen, we see that the coefficient of $b(\mathbf{x})$ is in $\mathbb{k}[c_1, \dots, c_n]$ and \mathbb{L} is a subfield of $\mathbb{k}[c_1, \dots, c_n]$. \square

We can strengthen Theorem 7.5 as follows:

Theorem 7.8. *Let $\alpha \in \mathbb{R}_{>0}^n$ and let $P(Z) \in \mathbb{k}[[\mathbf{x}]][[Z]]$ be a monic polynomial such that its discriminant $\Delta = \delta u$ where $\delta \in \mathbb{k}[[\mathbf{x}]]$ is (α) -homogeneous and $u \in \mathbb{k}[[\mathbf{x}]]$ is a unit. Let us set $N := \dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n)$. Then there exist $\gamma_1, \dots, \gamma_N$ integral homogeneous elements with respect to ν_α and a (α) -homogeneous polynomial $c(\mathbf{x}) \in \mathbb{k}[[\mathbf{x}]]$ such that the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\mathbb{k}'[[\mathbf{x}]][\gamma_1, \dots, \gamma_N]$ where $\mathbb{k} \rightarrow \mathbb{k}'$ is finite.*

Proof. If $Q(Z)$ is a monic polynomial dividing $P(Z)$ in $\mathbb{k}[[\mathbf{x}]][[Z]]$, then the discriminant of $Q(Z)$ divides the discriminant of $P(Z)$. Thus we may assume that $P(Z)$ is irreducible.

We will consider three cases: first the case where the coefficients of $P(Z)$ are complex analytic with $\alpha \in \mathbb{N}^n$, then with $\alpha \in \mathbb{R}_{>0}^n$, and finally the general case.

• Let us assume that $\alpha \in \mathbb{N}^n$ and that $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$. By Theorem 5.6 the roots of $P(Z)$ are of the form

$$\sum_{i_1, \dots, i_s} A_{i_1, \dots, i_s} \gamma_1^{i_1} \dots \gamma_s^{i_s}$$

where $\gamma_1, \dots, \gamma_s$ are integral homogeneous elements with respect to ν_α and $A_{i_1, \dots, i_s} \in \mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$ for any i_1, \dots, i_s . We may even choose $s = 1$ by Proposition 3.25, but we treat here the general case $s \geq 1$ that will be used in the sequel.

For $1 \leq i \leq s$, let q_i be the degree of the minimal polynomial of γ_i over $\mathbb{K}_n[\gamma_i, \dots, \gamma_{i-1}]$, and for $1 \leq i \leq s$, let $\gamma_{i,1}, \dots, \gamma_{i,q_i}$ be the conjugates of $\gamma_i = \gamma_{i,1}$ over $\mathbb{K}_n[\gamma_1, \dots, \gamma_{i-1}]$. Thus, by Proposition 7.3 and Remark 7.6, the roots of $P(Z)$ are of the form

$$\begin{aligned} & \sum_{\substack{0 \leq i_1 < q_1 \\ \dots \\ 0 \leq i_s < q_s}} A_{i_1, \dots, i_s} \gamma_{1,j_1}^{i_1} \dots \gamma_{s,j_s}^{i_s} \end{aligned}$$

where $A_{i_1, \dots, i_s} \in \mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$, $\nu_\alpha(A_{i_1, \dots, i_s} \gamma_{1,j_1}^{i_1} \dots \gamma_{s,j_s}^{i_s}) \geq 0$ for any i_1, \dots, i_s , and $1 \leq j_i \leq q_i$ for any i .

Let us assume that $P(Z)$ factors into a product of monic irreducible polynomials as $P(Z) =$

$P_1(Z) \dots P_r(Z)$ in $\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1 \leq i < s, 0 \leq j < q_i}[Z]$. Let us write the roots of $P_1(Z)$ as $z_j = \sum_{i=0}^{q_s-1} B_i \gamma_{s,j}^i$

where $B_i \in \mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1 \leq i < s, 0 \leq j < q_i}$ for all i . Then the roots of the other $P_j(Z)$'s are

$\sum_{i=0}^{q_s-1} B'_i \gamma_{s,j}^i$ where $(B'_0, \dots, B'_{q_s-1})$ is the image (B_0, \dots, B_{q_s-1}) by a $\mathcal{K}_\alpha^{\mathbb{C}\{x\}}$ -automorphism of

$\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1 \leq i < s, 0 \leq j < q_i}$. If the roots of $P_1(Z)$ satisfy the theorem, then we see that the roots of the other $P_j(Z)$'s will also satisfy the theorem since they are conjugates of the roots of $P_1(Z)$ by $\mathcal{K}_\alpha^{\mathbb{C}\{x\}}$ -automorphisms of $\mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}[\gamma_{i,j}]_{1 \leq i < s, 0 \leq j < q_i}$. Thus it is enough to prove the result for the roots of $P_1(Z)$. We have

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{q_s} \end{pmatrix} = \begin{pmatrix} 1 & \gamma_{s,1} & \dots & \gamma_{s,1}^{q_s-1} \\ 1 & \gamma_{s,2} & \dots & \gamma_{s,2}^{q_s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_{s,q_s} & \dots & \gamma_{s,q_s}^{q_s-1} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_{q_s-1} \end{pmatrix}.$$

Let us define $M := \begin{pmatrix} 1 & \gamma_{s,1} & \dots & \gamma_{s,1}^{q_s-1} \\ 1 & \gamma_{s,2} & \dots & \gamma_{s,2}^{q_s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \gamma_{s,q_s} & \dots & \gamma_{s,q_s}^{q_s-1} \end{pmatrix}$. The determinant of M is a homogeneous

element c with respect to ν_α where $\nu_\alpha(c) = \frac{1}{2}q_s(q_s - 1)\nu_\alpha(\gamma_s)$. Thus we have

$$B_i = \frac{1}{c} (R_{i,1}(\gamma_{s,1}, \dots, \gamma_{s,q_s})z_1 + \dots + R_{i,s}(\gamma_{s,1}, \dots, \gamma_{s,q_s})z_s)$$

where the $R_{i,j}$'s are polynomials with coefficients in \mathbb{Q} and $R_{i,j}(\gamma_{s,1}, \dots, \gamma_{s,q_s})$ is homogeneous with respect to ν_α . By multiplying c and $R_{i,1}(\gamma_{s,1}, \dots, \gamma_{s,q_s})z_1 + \dots + R_{i,s}(\gamma_{s,1}, \dots, \gamma_{s,q_s})z_s$ by

the conjugates of c over $\mathbb{k}[\mathbf{x}]$ we may assume that $c = c(\mathbf{x}) \in \mathbb{k}[\mathbf{x}]$ is a (α) -homogeneous polynomial. The z_i 's and the $\gamma_{s,j}$'s are locally analytic on $D_{\theta,\eta} := B(0,\eta) \setminus \theta^{-1}(0)$ and bounded near the origin, where $\{\theta = 0\}$ contains the discriminant locus of $P(Z)$ and of the minimal polynomials of the γ_i 's and η is small enough. Thus $c(\mathbf{x})B_i$ is locally analytic on $D_{\theta,\eta}$ for $1 \leq i \leq q_s$ and is bounded near the origin. Moreover $c(\mathbf{x})B_i$ is algebraic over \mathcal{F}_n since the $g_{s,j}$'s and the z_k 's are algebraic over \mathcal{F}_n . By induction on s (we replace z_1, \dots, z_{q_s} by $c(\mathbf{x})B_0, \dots, c(\mathbf{x})B_{q_s-1}$) we see that there exists a (α) -homogeneous polynomial $c(\mathbf{x})$ such that $c(\mathbf{x})A_{\underline{i}}$ is locally analytic on $D_{\theta,\eta}$ and bounded near the origin for any $\underline{i} := (i_1, \dots, i_s)$. Since $c(\mathbf{x})A_{\underline{i}} \in \mathcal{K}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$ and it is bounded near the origin, we see that $c(\mathbf{x})A_{\underline{i}} \in \mathcal{V}_\alpha^{\mathbb{C}\{\mathbf{x}\}}$. Thus it is analytic on $\mathcal{D}_{\theta,C,a,\eta}$ for C, a and η well chosen (see Lemma 7.2). Hence by Lemma 7.4 it is analytic on $D_{\theta,\eta}$ and since it is bounded near the origin, $c(\mathbf{x})A_{\underline{i}} \in \mathbb{C}\{\mathbf{x}\}$ for any \underline{i} .

- Now let us consider any $\alpha \in \mathbb{R}_{>0}^n$. Then the roots of $P(Z)$ are in $\mathcal{V}_\alpha[\langle \gamma_1, \dots, \gamma_s \rangle]$ for some integral homogeneous elements with respect to ν_α denoted by $\gamma_1, \dots, \gamma_s$. Let us denote these roots by z_1, \dots, z_d . For any $\alpha' \in \mathbb{N}^n$ such that $\text{Rel}_\alpha \subset \text{Rel}_{\alpha'}$, $\gamma_1, \dots, \gamma_s$ are integral homogeneous elements with respect to $\nu_{\alpha'}$. Thus, for any $\varepsilon > 0$ small enough (say $\varepsilon < \varepsilon_0$), for any $q \in \mathbb{N}$ and any $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$, $z_1, \dots, z_d \in \mathcal{V}_{\alpha'}[\langle \gamma_1, \dots, \gamma_s \rangle]$ by Proposition 6.7. Moreover, by the previous case, we see that

$$\forall \varepsilon \in]0, \varepsilon_0[, \forall q \in \mathbb{N}, \forall \alpha' \in \text{Rel}(\alpha, q, \varepsilon), \exists c_{\alpha'}(\mathbf{x}) \text{ an } (\alpha')\text{-homogeneous polynomial such that}$$

$$c_{\alpha'}(\mathbf{x})z_1, \dots, c_{\alpha'}(\mathbf{x})z_d \in \mathbb{C}\{\mathbf{x}\}[\gamma_1, \dots, \gamma_s].$$

Moreover we see that that $c_\alpha(\mathbf{x})$ may be chosen as being the product of the determinants of Vandermonde matrices as M depending only on $\gamma_1, \dots, \gamma_s$, thus $c_{\alpha'}(\mathbf{x})$ does not depend on α' . Let us denote $c(\mathbf{x}) := c_{\alpha'}(\mathbf{x})$. Since $c(\mathbf{x})$ is a (α') -homogeneous polynomial for all $\alpha' \in \text{Rel}(\alpha, q, \varepsilon)$ then $c(\mathbf{x})$ is a (α) -homogeneous polynomial. This proves the result.

- Now let us consider the general case, $\alpha \in \mathbb{R}_{>0}^n$ and $P(Z) \in \mathbb{k}[\![\mathbf{x}]\!][Z]$ where \mathbb{k} is a field of characteristic zero.

Let us write $P(Z) = Z^d + a_{d-1}(\mathbf{x})Z^{d-1} + \dots + a_0(\mathbf{x})$. Exactly as in the proof of Theorem 7.5 we may assume that \mathbb{C} is a field extension of \mathbb{k} and $P(Z) \in \mathbb{C}[\![\mathbf{x}]\!]$. Let us use the notation of Theorem 7.5. Let

$$R(A_0, \dots, A_{d-1}, U) := D(A_0, \dots, A_{d-1}) - \delta(\mathbf{x})U \in \mathbb{C}[\mathbf{x}][A_0, \dots, A_{d-1}, U]$$

where D is the universal discriminant of a monic polynomial of degree d . Then

$$R(a_0(\mathbf{x}), \dots, a_{d-1}(\mathbf{x}), u(\mathbf{x})) = 0.$$

By Artin Approximation Theorem [Art], for any integer $c > 0$, there exist convergent power series $\bar{a}_{0,c}(\mathbf{x}), \dots, \bar{a}_{d-1,c}(\mathbf{x}), \bar{u}_c(\mathbf{x}) \in \mathbb{C}\{\mathbf{x}\}$ such that

$$(12) \quad R(\bar{a}_{0,c}(\mathbf{x}), \dots, \bar{a}_{d-1,c}(\mathbf{x}), \bar{u}_c(\mathbf{x})) = 0,$$

and

$$\bar{a}_{k,c}(\mathbf{x}) - a_k(\mathbf{x}), \bar{u}_c(\mathbf{x}) - u(\mathbf{x}) \in (\mathbf{x})^c \quad \text{for } 0 \leq k \leq d.$$

Let $P_{(c)}(Z) := Z^d + \bar{a}_{d-1,c}(\mathbf{x})Z^{d-1} + \dots + \bar{a}_{0,c}(\mathbf{x})$. Then $P_{(c)}(Z)$ is irreducible for c large enough (say $c \geq c_0$). Moreover the discriminant of $P_{(c)}(Z)$ is of the form $\delta(\mathbf{x})u_{(c)}$ where $u_{(c)}$ is a unit in $\mathbb{C}\{\mathbf{x}\}$ if $c \geq 1$ by Equation (10). By the previous case, the roots of $P_{(c)}(Z)$ are in $\frac{1}{c_c(\mathbf{x})}\mathbb{C}\{\mathbf{x}\}[\gamma_{1,c}, \dots, \gamma_{N,c}]$ where $\gamma_{1,c}, \dots, \gamma_{N,c}$ are integral homogeneous elements with respect to ν_α and $c_c(\mathbf{x})$ is a (α) -homogeneous polynomial. By Proposition 4.14 and the previous cases, we may assume that the $\gamma_{i,c}$ does not depend on c , thus let us denote $\gamma_{i,c}$ by γ_i . Moreover $c_c(\mathbf{x})$ may be chosen as being the product of the determinants of Vandermonde

matrices as M depending only on $\gamma_1, \dots, \gamma_s$, thus $c_c(\mathbf{x})$ does not depend on c . Let us denote by $c(\mathbf{x})$ this common (α) -homogeneous polynomial.

Thus, when c goes to infinity, we see that the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\mathbb{C}[[\mathbf{x}]][\gamma_1, \dots, \gamma_N]$. Such a root has the form $\sum_{i_1, \dots, i_N} A_{i_1, \dots, i_N} \gamma_1^{i_1} \dots \gamma_N^{i_N}$ where i_k runs from 0 to $q_k - 1$. In this case $c(\mathbf{x})A_{i_1, \dots, i_N} \in \mathbb{C}[[\mathbf{x}]]$ is algebraic over $\mathbb{k}[[\mathbf{x}]]$, thus $c(\mathbf{x})A_{i_1, \dots, i_N} \in \mathbb{k}'[[\mathbf{x}]]$ where $\mathbb{k} \rightarrow \mathbb{k}'$ is finite by Proposition 7.7. Thus the roots of $P(Z)$ are in $\frac{1}{c(\mathbf{x})}\mathbb{k}'[[\mathbf{x}]][\gamma_1, \dots, \gamma_N]$. \square

In the case the α_i 's are linearly independent over \mathbb{Q} , we can choose $c(\mathbf{x}) = 1$. This is exactly the Abhyankar-Jung Theorem:

Corollary 7.9 (Abhyankar-Jung Theorem). *Let $P(Z) \in \mathcal{F}_n(Z)$ be a monic polynomial whose discriminant has the form $\mathbf{x}^\beta u(\mathbf{x})$ where $\beta \in \mathbb{N}^n$ and $u(0) \neq 0$. Then there exists an integer $q \in \mathbb{N}$ and a finite field extension $\mathbb{k} \rightarrow \mathbb{k}'$ such that the roots of $P(Z)$ are in $\mathbb{k}'[[x_1^{\frac{1}{q}}, \dots, x_n^{\frac{1}{q}}]]$.*

Proof. By the previous theorem, where $\alpha \in \mathbb{R}_{\geq 0}^n$ satisfies $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$, the roots of $P(Z)$ are in $\frac{1}{\mathbf{x}^\gamma}\mathbb{k}'[[x_1^{\frac{1}{q}}, \dots, x_n^{\frac{1}{q}}]]$ for some $\beta \in \mathbb{N}^n$, $q \in \mathbb{N}$ and $\mathbb{k} \rightarrow \mathbb{k}'$ a finite field extension. Since the discriminant of any monic factor of $P(Z)$ in $\mathbb{k}'[[x_1, \dots, x_n]][Z]$ divides the discriminant of $P(Z)$, we may assume that $P(Z)$ is irreducible in $\mathbb{k}'[[x_1, \dots, x_n]][Z]$, thus we assume that $\mathbb{k}' = \mathbb{k}$.

Let z be a root of $P(Z)$ and let us denote by $\text{NP}(z)$ its Newton polyhedron. Then $\text{NP}(z) \subset -\gamma + \mathbb{R}_{\geq 0}^n$ and $\langle \alpha, \gamma' \rangle \geq 0$ for any $\gamma' \in \text{NP}(z)$. Let us assume that $\text{NP}(z) \not\subset \mathbb{R}_{\geq 0}^n$. This means that there exists $\gamma' \in \text{NP}(z)$ such that one its coordinates, let us say γ'_n , is negative. For any $t \in [1, +\infty[$ let us set:

$$\begin{aligned} \alpha_t &:= (\alpha_1, \dots, \alpha_{n-1}, t\alpha_n), \\ L_t &:= \{y \in \mathbb{R}^n \mid \langle \alpha_t, y \rangle = 0\}, \\ H_t &:= \{y \in \mathbb{R}^n \mid \langle \alpha_t, y \rangle \geq 0\}. \end{aligned}$$

Then $\text{NP}(z) \subset H_1$ and $\text{NP}(z) \not\subset H_t$ for t large enough. Thus there exists $t_0 \in [1, +\infty[$ such that $\text{NP}(z) \subset H_{t_0}$ and $\text{NP}(z) \cap L_{t_0} \neq \{0\}$. Let us write $z = \sum_{i \geq 0} z_i$ where z_i is a (α_{t_0}) -homogeneous rational fraction with $\nu_{\alpha_{t_0}}(z_i) = i$ for any i . By assumption $z_0 \notin \mathbb{k}$. Let $\xi = (\xi_1, \dots, \xi_n)$ be a vector of q -th roots of unity and let us denote by z_ξ the element $z(\xi_1 x_1, \dots, \xi_n x_n)$. Then the roots of $P(Z)$ are the z_ξ . In particular the constant coefficient $a(\mathbf{x})$ of $P(Z)$ is a product of some of these z_ξ (not all of them since it may happen that $z_\xi = z_{\xi'}$ for $\xi \neq \xi'$): $a(\mathbf{x}) = \prod_{\xi \in I} z_\xi$ where I is finite set. We have $z_\xi = \sum_i z_{\xi, i}$ where $z_{\xi, i} = z_i(\xi_1 x_1, \dots, \xi_n x_n)$. If we write $a(\mathbf{x}) = \sum_i a_i(\mathbf{x})$ where $a_i(\mathbf{x})$ is a (α_{t_0}) -homogeneous polynomial, we have $a_0(\mathbf{x}) = \prod_{\xi \in I} z_{\xi, 0}$. Thus the Newton polyhedron of $a(\mathbf{x})$ (i.e. the convex hull of the exponents of non-zero monomials of $a(\mathbf{x})$) is the Minkowski sum of the Newton polyhedra of the $z_{\xi, 0}$'s when $\xi \in I$ (see [Kh] for instance). Thus the Newton polyhedron of $a(\mathbf{x})$ is k times the Newton polyhedron of z_0 , where k is the cardinal of I , since the Newton polyhedra of the $z_{\xi, 0}$'s are equal. Thus the Newton polyhedron of $a(\mathbf{x})$ is not included in $\mathbb{R}_{\geq 0}^n$ since the Newton polyhedron of z_0 is not in $\mathbb{R}_{\geq 0}^n$ (since $z_0 \notin \mathbb{k}$). This is a contradiction, hence $\text{NP}(z) \subset \mathbb{R}_{\geq 0}^n$. \square

Let us finish this part by giving few results which are analogue to the fact that if $z \in \mathbb{C}\{t^{\frac{1}{k}}\}$ for some $k \in \mathbb{N}$, t being a single variable, then its minimal polynomial over $\mathbb{C}[[t]]$ is a polynomial with convergent power series:

Corollary 7.10. *Let $P(Z) \in \mathcal{F}_n(Z)$ be an irreducible monic polynomial such that the discriminant of $P(Z)$ has the form $\delta(\mathbf{x})u(\mathbf{x})$, where $\delta(\mathbf{x})$ is a (α) -homogeneous polynomial, $\alpha \in \mathbb{R}_{>0}^n$, and $u(\mathbf{x}) \in \mathcal{F}_n$ is invertible. Let us assume that $P(Z)$ has a root in $\mathcal{V}_\alpha^R[\langle \gamma_1, \dots, \gamma_s \rangle]$ where R is an excellent Henselian local ring satisfying Properties (A), (B) and (C) and $\gamma_1, \dots, \gamma_s$ are homogeneous elements with respect to ν_α . Then the coefficients of $P(Z)$ are in R .*

Proof. By Theorem 7.5, $P(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. Let $z \in \mathcal{V}_\alpha^R[\langle \gamma_1, \dots, \gamma_s \rangle]$ be a root of $P(Z)$ as given in the statement. We can write $z = \sum A_{i_1, \dots, i_s} \gamma_1^{i_1} \dots \gamma_s^{i_s}$ where the sum is finite and $A_{i_1, \dots, i_s} \in \mathcal{V}_\alpha^R$. Then the others roots of $P(Z)$ are of the form $\sum A_{i_1, \dots, i_s} \sigma(\gamma_1)^{i_1} \dots \sigma(\gamma_s)^{i_s}$ where σ is a $\mathbb{K}_{\nu_\alpha}^{\text{alg}}$ -automorphism of $\overline{\mathbb{K}_\nu}^{\text{alg}}$. Thus all the roots of $P(Z)$ are in $\overline{\mathcal{V}_\alpha^R}$. Hence the coefficients of $P(Z)$ are in $\overline{\mathcal{V}_\alpha^R} \cap \mathcal{F}_n = R$. \square

Definition 7.11. Let \mathbb{k} be a valued field and let σ be a strictly convex rational cone of \mathbb{R}^n containing $\mathbb{R}_{\geq 0}^n$. There exists an invertible $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ such that $M\gamma \in \mathbb{R}_{\geq 0}^n$ for any $\gamma \in \sigma$. We denote by $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \mathbb{Z}^n\}$ the subring of $\mathbb{k}[[\mathbf{x}^\beta, \beta \in \sigma \cap \mathbb{Z}^n]]$ of power series $f(\mathbf{x})$ such that $f(\tau(\mathbf{x})) \in \mathbb{k}\{\mathbf{x}\}$ where τ is the map defined by

$$(\tau(x_1), \dots, \tau(x_n)) = (x_1^{m_{1,1}} \dots x_n^{m_{1,n}}, \dots, x_1^{m_{n,1}} \dots x_n^{m_{n,n}}).$$

By Example 6.13 $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \mathbb{Z}^n\}$ is a subring of $\mathcal{V}_{\alpha, \delta}^{\mathbb{k}\{x\}}$ for any α such that $\langle \alpha, \gamma \rangle > 0$ for all $\gamma \in \sigma \setminus \{0\}$.

Let us mention the following theorem proven by A. Gabrielov and J.-Cl. Tougeron using transcendental methods (they use in a crucial way the maximum principle for analytic functions):

Theorem 7.12. [Ga][To] *Let $P(Z) \in \mathbb{C}[[\mathbf{x}]](Z)$ be an irreducible monic polynomial. If one root of $P(Z)$ is in $\mathbb{C}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ where σ is a strictly convex rational cone and $q \in \mathbb{N}$, then $P(Z) \in \mathbb{C}\{\mathbf{x}\}[Z]$.*

Using what we have done we can prove a particular case of this theorem over any algebraically closed valued field of characteristic zero. First we need the following lemma:

Lemma 7.13. *Let $P(Z) \in \mathbb{k}[[\mathbf{x}]](Z)$ be an irreducible monic polynomial where \mathbb{k} is a characteristic zero algebraically closed valued field. Let $\alpha \in \mathbb{R}_{>0}^n$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$ and $P(Z)$ is irreducible in $\mathcal{V}_\alpha[Z]$. By Theorem 6.9, the roots of $P(Z)$ are in $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ where σ is a strictly convex rational cone such that $\langle \alpha, \gamma \rangle > 0$ for any $\gamma \in \sigma$, $\gamma \neq 0$, and $q \in \mathbb{N}$. If one root of $P(Z)$ is in $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$, then the others roots of $P(Z)$ are in $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ and $P(Z) \in \mathbb{k}\{\mathbf{x}\}[Z]$.*

Proof. Let $z \in \mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ be a root of $P(Z)$. For any $\xi = (\xi_1, \dots, \xi_n)$ vector of q -th roots of unity let us denote by z_ξ the element of $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ obtain from z by replacing $(x_1^{\frac{1}{q}}, \dots, x_n^{\frac{1}{q}})$ by $(\xi_1 x_1^{\frac{1}{q}}, \dots, \xi_n x_n^{\frac{1}{q}})$. In particular $z_\xi \in \mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$. Then for any ξ , z_ξ is a root of $P(Z)$. Let I be a subset of \mathbb{U}_q^n , where \mathbb{U}_q is the group of q -th root of unity, such that

$$\begin{aligned} z_\xi &\neq z_{\xi'} \text{ for any } \xi, \xi' \in I, \xi \neq \xi', \\ \text{and } \forall \xi \in \mathbb{U}_q^n, \exists \xi' \in I, z_{\xi'} &= z_\xi. \end{aligned}$$

Let us set $Q(Z) = \prod_{\xi \in I} (Z - z_\xi)$. Then $Q(Z)$ is a monic polynomial of $\mathcal{V}_\alpha[Z]$ whose roots are roots of $P(Z)$. Thus it divides $P(Z)$ in $\mathcal{V}_\alpha[Z]$ hence, since $P(Z)$ is irreducible, $Q(Z) = P(Z)$. Thus the other roots of $P(Z)$ are in $\mathbb{k}\{\mathbf{x}^\beta, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ and $P(Z) \in \mathbb{k}\{\mathbf{x}\}[Z]$. \square

Corollary 7.14. *Let $P(Z) \in \mathbb{k}[[\mathbf{x}]](Z)$ be an irreducible monic polynomial of degree d where \mathbb{k} is a characteristic zero algebraically closed valued field. Let $\alpha \in \mathbb{R}_{>0}^n$ such that $\dim_{\mathbb{Q}}(\mathbb{Q}\alpha_1 + \dots + \mathbb{Q}\alpha_n) = n$. Let us assume that there exists an irreducible monic polynomial $Q(Z) \in \mathbb{k}[[\mathbf{x}]](Z)$ of degree d whose discriminant Δ_Q is a monomial times a unit such that*

$$\nu_{\alpha}(P(Z) - Q(Z)) \geq \frac{d}{2}\nu_{\alpha}(\Delta_Q).$$

Let us assume that one of the roots of $P(Z)$ is in $\mathbb{k}\{\mathbf{x}^{\beta}, \beta \in \sigma \cap \frac{1}{q}\mathbb{Z}^n\}$ for some strictly convex rational cone σ , where $\langle \alpha, \gamma \rangle > 0$ for any $\gamma \in \sigma \setminus \{0\}$, and $q \in \mathbb{N}$. Then the coefficients of $P(Z)$ are in $\mathbb{k}\{\mathbf{x}\}$.

Proof. By Remark 4.12 and Proposition 4.11, the polynomial $Q(Z)$ is irreducible in $\mathcal{V}_{\alpha}[Z]$. Thus we apply the previous Lemma. \square

8. DIOPHANTINE APPROXIMATION

Here we give a necessary condition for an element of $\widehat{\mathbb{K}}_{\nu}$ to be algebraic over \mathbb{K}_n :

Theorem 8.1. [Ro1][II] *Let ν be an Abhyankar valuation and let $z \in \mathbb{K}_{\nu}^{alg}$. Then there exist two constants $C > 0$ and $a \geq 1$ such that*

$$\left| z - \frac{f}{g} \right|_{\nu} \geq C|g|_{\nu}^a \quad \forall f, g \in \mathcal{F}_n.$$

Proof. Let $P(Z) := a_0Z^d + a_1Z^{d-1} + \dots + a_d \in \mathbb{K}_n[Z]$ be an irreducible polynomial such that $P(z) = 0$. Let $h \in \mathcal{F}_n$ and set

$$P_h(Z) := h^d a_0^{d-1} P\left(\frac{Z}{ha_d}\right).$$

Then $P_h(Z) = Z^d + a_1hZ^{d-1} + a_2a_0h^2Z^{d-2} + \dots + a_da_0^{d-1}h^d$ and zha_d is a root of $P_h(Z)$. It is straightforward to check that z satisfies the theorem if and only if zha_d does. Thus we may assume that $P(Z)$ is a monic polynomial and $\nu(z) > 0$ by choosing h such that $\nu(h)$ is large enough. Let us set $Q(Z_1, Z_2) := Z_1^d P(Z_2/Z_1)$. By Theorem 3.1 [Ro1] there exist two constants $a \geq d$ and $b \geq 0$ such that

$$\text{ord}(Q(f, g)) \leq a \min\{\text{ord}(f), \text{ord}(g)\} + b \quad \forall f, g \in \mathcal{F}_n.$$

Moreover, by Izumi's Theorem ([Iz], [Re], [ELS]), there exists a constant $c \geq 1$ such that for all $f \in \mathcal{F}_n$, $\text{ord}(f) \leq \nu(f) \leq c \text{ord}(f)$. Thus

$$\nu(Q(f, g)) \leq ac \min\{\nu(f), \nu(g)\} + bc \quad \forall f, g \in \mathcal{F}_n.$$

Since $P(Z)$ is irreducible in $\mathbb{K}_n[Z]$ and \mathbb{K}_n is a characteristic zero field, $P(Z)$ has no multiple roots in \widehat{V}_{ν} and we may write

$$P(Z) = R(Z)(Z - z)$$

where $R(Z) \in \widehat{V}_{\nu}[Z]$ and $R(z) \neq 0$. Set $r := \nu(z)$. Let $f, g \in \mathcal{F}_n$ with $g \neq 0$. Two cases may occur: either

$$(13) \quad \left| z - \frac{f}{g} \right|_{\nu} \geq e^{-r}$$

either $\nu\left(z - \frac{f}{g}\right) > r$. In the last case we have $\nu\left(\frac{f}{g}\right) = \nu(z) > 0$. In particular $\nu\left(R\left(\frac{f}{g}\right)\right) \geq 0$ and $\nu(f) > \nu(g)$. Thus

$$(ac - d)\nu(g) + bc \geq \nu\left(P\left(\frac{f}{g}\right)\right) \geq \nu\left(\frac{f}{g} - z\right).$$

Thus we have

$$(14) \quad A\nu(g) + B \geq \nu\left(\frac{f}{g} - z\right) \quad \text{or} \quad \left|z - \frac{f}{g}\right|_\nu \geq e^{-B}|g|_\nu$$

with $A = ac - d$ and $B = bc$. Then (13) and (14) prove the theorem. \square

Example 8.2. Let $\sigma := (-1, 1)\mathbb{R}_{\geq 0} + (1, 0)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. This is a rational strictly convex cone of \mathbb{R}^2 . Let $f(x_1, x_2)$ be a power series, $f(x_1, x_2) \in \mathbb{k}[[x_1, x_2]]$. Let us set

$$g(x_1, x_2) := \sum_{i=0}^{\infty} \left(\frac{x_2}{x_1}\right)^{i!} + f(x_1, x_2) \in \mathbb{k}[[x^\beta, \beta \in \sigma \cap \mathbb{Z}]].$$

Then $g \in \mathcal{V}_\alpha$ for any $\alpha \in \mathbb{R}_{>0}^2$ such that $\alpha_2 > \alpha_1$. Moreover

$$\nu_\alpha\left(g - f - \sum_{i=0}^n \left(\frac{x_2}{x_1}\right)^{i!}\right) = (n+1)!(\alpha_2 - \alpha_1) = \frac{\alpha_2 - \alpha_1}{\alpha_1}(n+1)\nu_\alpha(x_1^{n!}).$$

Thus there do not exist constants A and B such that

$$A\nu_\alpha(x_1^{n!}) + B \geq \nu_\alpha\left(g - f - \sum_{i=0}^n \left(\frac{x_2}{x_1}\right)^{i!}\right) \quad \forall n \in \mathbb{N}.$$

Hence $g(x_1, x_2)$ is not algebraic over \mathcal{F}_2 by Theorem 8.1.

NOTATIONS

- $\mathcal{F}_n := \mathbb{k}[[x_1, \dots, x_n]]$ is the ring of formal power series over \mathbb{k} .
- ν_α is the monomial valuation defined by $\nu_\alpha(x_i) := \alpha_i$ for any i (cf. Example 2.4).
- V_ν is the valuation ring associated to ν .
- \widehat{V}_ν is the completion of V_ν .
- \mathbb{K}_n is the fraction field of \mathcal{F}_n and V_ν .
- $\widehat{\mathbb{K}}_\nu$ is the fraction field of \widehat{V}_ν .
- $\text{Gr}_\nu V_\nu$ is the graded ring associated to V_ν (cf. Part 3).
- V_ν^{alg} is the algebraic closure (or the Henselization) of V_ν in \widehat{V}_ν (see Lemma 2.9).
- $\mathbb{K}_\nu^{\text{alg}}$ is the fraction field of V_ν^{alg} .
- V_ν^{fg} is the subring of \widehat{V}_ν whose elements have ν -support included in a finitely generated sub-semigroup of $\mathbb{R}_{>0}$ (cf. Definition 3.11).
- $\mathbb{K}_\nu^{\text{fg}}$ is the fraction field of V_ν^{fg} .
- A (α) -homogeneous polynomial is a weighted homogeneous polynomial for the weights $\alpha_1, \dots, \alpha_n$ (see Definition 3.18).
- $A[\langle \gamma_1, \dots, \gamma_s \rangle]$ is the valuation ring associated to $A[\gamma_1, \dots, \gamma_s]$ when $A = \widehat{V}_\nu, V_\nu^{\text{fg}}$ or V_ν^{alg} (cf. Definition 3.23).
- \overline{V}_ν is the direct limit of the rings $\widehat{V}_\nu[\langle \gamma_1, \dots, \gamma_s \rangle]$ where the γ_i 's are homogeneous elements with respect to ν (cf. Definition 3.24).
- $\overline{\mathbb{K}}_\nu$ is the fraction field of \overline{V}_ν .
- $\overline{V}_\nu^{\text{alg}}$ is the direct limit of the rings $V_\nu^{\text{alg}}[\langle \gamma_1, \dots, \gamma_s \rangle]$ where the γ_i 's are homogeneous elements with respect to ν .
- $\overline{\mathbb{K}}_\nu^{\text{alg}}$ is the fraction field of $\overline{V}_\nu^{\text{alg}}$.
- $\overline{V}_\nu^{\text{fg}}$ is the direct limit of the rings $V_\nu^{\text{fg}}[\langle \gamma_1, \dots, \gamma_s \rangle]$ where the γ_i 's are homogeneous elements with respect to ν .

- $\overline{\mathbb{K}}_\nu^{\text{fg}}$ is the fraction field of $\overline{V}_\nu^{\text{fg}}$.
- $\mathcal{V}_{\alpha,\delta}$ is the subring of $V_{\nu_\alpha}^{\text{fg}}$ of elements of the form $\sum_{i \in \Lambda} \frac{a_i}{\delta^{m(i)}}$ where $\Lambda \subset \mathbb{R}$ is a finitely generated semigroup, $\nu_\alpha \left(\frac{a_i}{\delta^{m(i)}} \right) = i$ and $i \mapsto m(i)$ is bounded by an affine function (see Definition 5.1).
- \mathcal{V}_α is the direct limit of the $\mathcal{V}_{\alpha,\delta}$'s over all the (α) -homogeneous polynomials δ . It is a valuation ring (cf. Proposition 5.3).
- \mathcal{K}_α is the fraction field of \mathcal{V}_α (cf. Definition 5.4).
- $\overline{\mathcal{K}}_\alpha$ is the direct limit of the fields $\mathcal{K}[\langle \gamma_1, \dots, \gamma_s \rangle]$ where the γ_i 's are homogeneous elements with respect to ν (cf. Definition 5.4).
- $\mathcal{V}_{\alpha,\delta}^R$ is the subring $\mathcal{V}_{\alpha,\delta}$ whose elements are in the Henselian ring R after a suitable transform (cf. Definition 6.11).
- \mathcal{V}_α^R is the direct limit of the $\mathcal{V}_{\alpha,\delta}^R$ over all the (α) -homogeneous polynomials δ .

REFERENCES

- [Ab] S. Abhyankar, On the ramification of algebraic functions, *Amer. J. Math.*, **77**, (1955), 575-592.
- [AM] S. Abhyankar, T. Moh, On analytic independence, *Trans. A.M.S.*, **219**, (1976), 77-87.
- [Aro] F. Aroca, Puiseux parametric equations of analytic sets, *Proc. Amer. Math. Soc.*, **132**, (2004), no. 10, 3035-3045.
- [AI] F. Aroca, G. Ilardi, A family of algebraically closed fields containing polynomials in several variables, *Comm. Algebra*, **37**, (2009), no. 4, 1284-1296.
- [Art] M. Artin, On the solutions of analytic equations, *Invent. Math.*, **5**, (1968), 277-291.
- [BK] E. Brieskorn, H. Knörrer, Plane algebraic curves, Birkhäuser, (1986).
- [Cu] S. D. Cutkosky, Resolution of Singularities, *Graduate Studies in Mathematics*, **63**, American Math. Soc., (2004).
- [CK] S. D. Cutkosky, O. Kashcheyeva, Algebraic series and valuation rings over nonclosed fields, *J. of Pure and Applied Algebra*, **212**, (2008), 1996-2012.
- [ELS] L. Ein, R. Lazarsfeld, K. Smith, Uniform approximation of Abhyankar valuation ideals in smooth function fields, *Amer. J. Math.*, **125**, (2003), no. 2, 409-440.
- [FJ] C. Favre, M. Jonsson, The Valuation Tree, *Lecture Notes in Mathematics*, 1853. Springer-Verlag, Berlin, 2004. xiv+234 pp.
- [Ga] A. Gabrielov, Formal relations between analytic functions, *Izv. Akad. Nauk. SSSR*, **37**, 1056-1088, (1973).
- [Go] P. D. González Pérez, Singularités quasi-ordinaires toriques et polyèdre de Newton du discriminant, *Canad. J. Math.*, **52**, (2000), 348-368.
- [Gr] A. Grothendieck, Étude locale des schémas et des morphismes de schémas IV, *Inst. Hautes Études Sci. Publ. Math.*, No. 32, 1967.
- [HOV] F. Herrera Govantes, M. Olalla Acosta, J. L. Vicente Córdoba, Valuations in fields of power series, *Proceedings of the International Conference on Algebraic Geometry and Singularities (Sevilla, 2001)*, *Rev. Mat. Iberoamericana*, **19**, (2003), no. 2, 467-482.
- [II] H. Ito, S. Izumi, Diophantine inequality for equicharacteristic excellent Henselian local domains, *C. R. Math. Acad. Sci. Soc. R. Can.*, **30**, (2008), no. 2, 48-55.
- [Iz] S. Izumi, A measure of integrity for local analytic algebras, *Publ. RIMS, Kyoto Univ.*, **21**, (1985), 719-736.
- [Ju] H. E. W. Jung, Darstellung der Funktionen eines algebraischen Körpers zweier unabhängiger Veränderlichen x, y in der Umgebung einer Stelle $x = a, y = b$. *J. Reine Angew. Math.*, **133**, (1908), 289-314.
- [Kh] A. Khovanskii, Newton polytopes (algebra and geometry), *Theory of operators in function spaces*, (Kuybyshev, 1988), 202-221, Saratov. Gos. Univ., Kuibyshev. Filial, Kuybyshev, 1989.
- [KV] K. Kiyek, J. L. Vicente, On the Jung-Abhyankar theorem, *Arch. Math. (Basel)*, **83**, (2004), no. 2, 123-134.
- [McC] P. J. McCarthy, Algebraic extensions of fields, Chelsea Publishing Co., New York, (1976), ix+166 pp.
- [McD] J. McDonald, Fiber polytopes and fractional power series, *Journal of Pure and Applied Algebra*, **104**, (1995), 213-233.
- [M-B] L. Moret-Bailly, An extension of Greenberg's theorem to general valuation rings, *Manuscripta Math.*, **139**, (2012), no. 1-2, 153-166.

- [PR] A. Parusiński, G. Rond, The Abhyankar-Jung Theorem, *Journal of Algebra*, **365**, (2012), 29-41.
- [Po] D. Popescu, General Neron desingularisation and approximation, *Nagoya Math. J.*, **104**, (1986), 85-115.
- [Pu1] V. Puiseux, Recherches sur les fonctions algébriques, *J. Math. Pures Appl.*, **15**, (1850), 365-480.
- [Pu2] V. Puiseux, Nouvelles recherches sur les fonctions algébriques, *J. Math. Pures Appl.*, **16**, (1851), 228-240.
- [Re] D. Rees, Izumi's theorem, Commutative algebra (Berkeley, CA, 1987), 407-416, *Math. Sci. Res. Inst. Publ.*, **15**, (1989).
- [Ri] P. Ribenboim, Fields: algebraically closed and others, *Manuscripta Math.*, **75**, (1992), 115-150.
- [Ro1] G. Rond, Approximation diophantienne dans le corps des séries en plusieurs variables, *Ann. Institut Fourier*, vol. 56, no. 2, (2006), 299-308.
- [Ro2] G. Rond, Homomorphisms of local algebras in positive characteristic, *J. Algebra*, **322**, (2009), no. 12, 4382-4407.
- [Sa] A. Sathaye, Generalized Newton-Puiseux expansions and Abhyankar-Moh semigroup theorem, *Invent. Math.*, **74**, (1983), 149-157.
- [Sp1] M. Spivakovsky, Valuations in function fields of surfaces, *Amer. J. Math.*, **112**, (1990), no. 1, 107-156.
- [Sp2] M. Spivakovsky, A new proof of D. Popescu's theorem on smoothing of ring homomorphisms, *J. Amer. Math. Soc.*, **12** (1999), no. 2, 381-444.
- [SV] M. J. Soto, J. L. Vicente, The Newton procedure for several variables, *Linear Algebra Appl.*, **435**, (2011), no. 2, 255-269.
- [To] J.-Cl. Tougeron, Sur les racines d'un polynôme à coefficients séries formelles, *Real analytic and algebraic geometry (Trento 1988)*, 325-363, *Lectures Notes in Math.*, **1420**, (1990).

INSTITUT DE MATHÉMATIQUES DE LUMINY, CAMPUS DE LUMINY, CASE 907, 13288 MARSEILLE CEDEX 9, FRANCE

E-mail address: `rond@iml.univ-mrs.fr`